Journal of Statistical Physics, Vol. 124, No. 5, September 2006 (© 2006)
DOI: 10.1007/s10955-006-9176-x

Condensation in a Disordered Infinite-Range Hopping Bose–Hubbard Model

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Received March 8, 2006; accepted July 9, 2006
Published August 11, 2006

We study Bose–Einstein Condensation (BEC) in the Infinite-Range-Hopping Bose–Hubbard model with repulsive on-site particle interaction in the presence of an ergodic random single-site external potential with different distributions. We show that the model is exactly soluble even if the on-site interaction is random. We observe new phenomena: instead of *enhancement* of BEC for perfect bosons, for constant on-site repulsion and discrete distributions of the single-site potential there is *suppression* of BEC at certain *fractional* densities. We show that this suppression appears with increasing disorder. On the other hand, the suppression of BEC at integer densities observed in Bru and Dorlas (*J. Stat. Phys.* **113**:177–195, 2003) in the absence of a random potential, can disappear as the disorder increases. For a *continuous distribution* we prove that the BEC critical temperature decreases for small on-site repulsion while the BEC is suppressed at integer values of the density for large repulsion. Again, the threshold for this repulsion gets higher, when disorder increases.

KEY WORDS: Bose-Hubbard model, Bose condensation, random potential, disorder

1. INTRODUCTION

Lattice Bose-gas models were invented as an alternative way to understand continuous interacting boson systems including liquid Helium, see Ref. 1 and a very complete review.[(2)] But recent experiments with cold bosons in traps of three-dimensional optical lattice potentials show that lattice models are also relevant for

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describing the experimentally observed Mott *insulator-superfluid* (or condensate) phase transition.[3] In Ref. 4 and then in Ref. 5, this phenomenon was analyzed rigorously in the framework of the so-called *Bose–Hubbard* model.

The aim of the present paper is to study a *disordered* Bose–Hubbard model and in particular the influence of the *single-site potential* randomness on the Bose–Einstein condensate (BEC).

Notice that the first attempts to understand this influence go back to Refs. 6–8 for continuous Perfect Bose-Gases (PBG) in a random potential of *impurities*. For the rigorous solution of this problem, see Ref. 9. One of the principal result of Ref. 9 is that the randomness *enhances* the BEC. For example, the one-dimensional PBG has no BEC because of the high value of the one-particle density of states in the vicinity of the bottom of the spectrum above the ground state, making the integral for the critical particle density infinite. The presence of a non-negative homogeneous ergodic random potential modifies the one-particle density of states (due to the *Lifshitz tail*) in such a way that the integral for the critical density becomes finite. Hence, the one-dimensional PBG with random potential does manifest BEC. The nature of this BEC is close to what is known as the "*Bose-glass*" since it may be localized by the random potential.[10] This is of interest for experiments with liquid ^4He in random environments like Aerogel and Vycor glass.[11,12]

On the other hand, the nature and behaviour of the *lattice* BEC may be quite different. First of all, the lattice Laplacian and the Bose–Hubbard interaction produce a coexistence of the BEC (*superfluidity*) and the *Mott insulating phase* as well as domains of *incompressibility*, see e.g. Refs. 11 and 13 Adding disorder makes the corresponding models much more complicated. The physical arguments [11,13] show that the randomness may *suppress* the BEC (superfluidity) as well as the Mott phase in favour of the localized *Bose-glass* phase, but this is very sensitive to the choice of the random distribution.

Since there are very few rigorous results about the BEC in disordered systems, we consider here a *single-site* random version of the lattice *Infinite-Range Hopping* (IRH) Bose–Hubbard model, which in the non-random case has recently been studied in detail for all temperatures and chemical potentials in Ref. 4.

This paper is organized as follows. In Sec. 2, we define the lattice Laplacian for finite- and infinite-range hopping and recall the results about BEC for the free lattice Bose-gas. We then introduce a random on-site particle interaction and a random single-site potential and state our *main result (Theorem 2.1) about the existence of and an explicit formula for the pressure* for the IRH Bose–Hubbard model with these types of randomness. We prove the main theorem using the approximating Hamiltonian method.

In Sec. 3 we consider the pressure for extremal cases of *hard-core* and *perfect* bosons. We show that they are the limits of the IRH Bose–Hubbard model pressure when the on-site particle interaction tends respectively to $+\infty$ and to 0.

In Sec. 4, we analyse the phase diagram in the case of a non-random on-site particle interaction and a random single-site external potential. We distinguish a number of different cases. We start with *perfect* bosons and show that the randomness *enhances* BEC in this case, see Sec. 4.1. This is no longer true for interacting bosons. We study in Sec. 4.2 the phase diagram for a discrete distribution of independent identically distributed (i.i.d.) single-site potentials. We first consider the case of *Bernoulli* single-site potentials and then that of *trinomial* and *multinomial* discrete distributions.

For *hard-core bosons* (infinite on-site repulsion) we show that, in the case of a Bernoulli distribution, where the single-site potentials ε_x^ω are equal to a positive constant $\varepsilon > 0$ with probability p and equal to 0 with probability $1 - p$, in addition to the complete *suppression* of BEC at the extremal allowed densities $\rho = 0$ and $\rho = 1$, there is also suppression of the BEC at a new point $\rho = 1 - p$, if ε is large. We also prove that for large but *finite on-site repulsion* the suppression of BEC at *integer* densities persists, and also occurs for *fractional* values of the density $\rho = n - p$, $n = 1, 2, \ldots$, provided the Bernoulli potential amplitude is large enough. In fact we find that increasing the Bernoulli potential amplitude (disorder) decreases the critical BEC temperature in the vicinity of fractional values of the density, but increases it for integer values of the density. A similar phenomenon occurs also for (equiprobable) trinomial distributions, but now for densities $\rho = n/3$. Our numerical calculations demonstrate that a similar phenomenon should be true for a general multinomial distribution.

As illustration of a continuous distribution we study a homogeneous distribution with compact support in Sec. 4.3. Then, for hard-core bosons, we prove that complete suppression of BEC occurs *only* at the extremal allowed densities $\rho = 0$ and $\rho = 1$, while the suppression at integer values of the density is incomplete for a finite on-site repulsion. In particular we show that the critical BEC temperature gets *lower*, when one switches on disorder for (a small) on-site interaction, whereas it gets *higher* for perfect bosons. For large values of the on-site interaction the picture is similar to that for discrete distributions: increasing of disorder increases the critical BEC temperature in the vicinity of integer values of density but increases it for complimentary values of density.

In Sec. 5 we summarize and discuss our results.

2. MODEL AND MAIN THEOREM

2.1. Bose–Hubbard Models and a Variational Formula for the Pressure in the Case of Infinite-Range Hopping

For simplicity, we shall consider the Bose–Hubbard model only with *periodic boundary conditions*. So let $\Lambda := \{x \in \mathbb{Z}^d : -L_\alpha/2 \leq x_\alpha < L_\alpha/2, \alpha = 1, \ldots, d\}$ be a bounded rectangular domain of the cubic lattice $\mathbb{Z}^d$ wrapped onto a *torus*. Then the set $\Lambda^* := \{q_\alpha = 2\pi n/L_\alpha : n = 0, \pm 1, \pm 2, \ldots \pm (L_\alpha/2 - 1), L_\alpha/2,\ \alpha = 1, 2, \ldots d\}$ is *dual* to Λ with respect to Fourier transformation on the domain $\Lambda = L_1 \times L_2 \times \ldots \times L_d$ of volume $|\Lambda| = V$.

The standard *one-particle* Hilbert space for the set Λ can be taken as $\mathfrak{h}(\Lambda) := \mathbb{C}^\Lambda$ with the canonical basis $\{e_x\}_{x\in\Lambda}$, i.e. $e_x(y) = \delta_{x\,,\,y}$. Then for any element $u = \sum_{x\in\Lambda} u_x e_x \in \mathfrak{h}(\Lambda)$ the one-particle *kinetic-energy* (*hopping*) operator is defined by

$$(t_\Lambda u)(x) := \sum_{y\in\Lambda} t^\Lambda_{x\,,\,y}(u_x - u_y), \tag{2.1}$$

where

$$t^\Lambda_{x\,y} = \frac{1}{V}\sum_{q\in\Lambda^*} \hat{t}_q e^{iq(x-y)}, \tag{2.2}$$

is the *periodic extension* in domain Λ of a symmetric, translation invariant and *positive-definite* matrix, i.e.

$$\hat{t}_q = \sum_{y\in\Lambda} t^\Lambda_{0\,,\,y} e^{iqy} \geq 0. \tag{2.3}$$

Notice that functions $\{(\hat{e}_q)(y) := e^{iqy}/\sqrt{V}\}_{q\in\Lambda^*}$ also form a basis in $\mathfrak{h}(\Lambda)$, i.e. for any $u \in \mathfrak{h}(\Lambda)$ one has $u = \sum_{q\in\Lambda^*} u_q \hat{e}_q$.

Let $\mathfrak{F}_B(\Lambda) := \mathfrak{F}_B(\mathfrak{h}(\Lambda))$ be the *boson Fock space* over $\mathfrak{h}(\Lambda)$. For any $f \in \mathfrak{h}(\Lambda))$ we can associate in this space the creation and annihilation operators

$$a^*(f) := \sum_{y\in\Lambda} a^*_y f(y), \qquad a(f) := \sum_{y\in\Lambda} a_y f^*(y). \tag{2.4}$$

By (2.4) we obtain: $a^*_x = a(e_x)^*$, $a_x = a(e_x)$ and $\hat{a}^*_q = a^*(\hat{e}_q)$, $\hat{a}_q := a(\hat{e}_q)$, for the boson creation and annihilation operators corresponding respectively to the basis elements e_x and $\hat{e}_q$. They satisfy the lattice *Canonical Commutation Relations* (CCR): $\left[a_x, a^*_y\right] = \delta_{x\,,\,y}$ and $\left[\hat{a}_q, \hat{a}^*_p\right] = \delta_{q\,,\,p}$. Then $n_x = a^*_x a_x$ is the *one-site* number operator, and

$$N_\Lambda := \sum_{x\in\Lambda} n_x = \sum_{q\in\Lambda^*} \hat{a}^*_q \hat{a}_q\,, \tag{2.5}$$

is the *total* number operator.

The second quantization of the hopping operator (2.1) in $\mathfrak{F}_B(\Lambda)$ gives the *free boson* Hamilton of the form

$$T_\Lambda := \sum_{x\,\in\Lambda} a^*_x(t_\Lambda a)_x = \frac{1}{2}\sum_{x,y\,\in\Lambda} t^\Lambda_{x\,y}(a^*_x - a^*_y)(a_x - a_y) = \sum_{q\in\Lambda^*}(\hat{t}_0 - \hat{t}_q)\hat{a}^*_q\hat{a}_q. \tag{2.6}$$

If hopping is allowed only between the *nearest neighbor* (n.n.) sites with equal probabilities, then $t_\Lambda = -\Delta$ corresponds to minus the *lattice Laplacian*, i.e.

$$t^{\Lambda}_{x\,y} = \sum_{\alpha=1}^{d} (\delta_{x+1_\alpha\,,\,y} + \delta_{x-1_\alpha\,,\,y}), \tag{2.7}$$

where $(x \pm 1_\alpha)_\beta = x_\beta \pm \delta_{\alpha\,,\,\beta}$. In this case the one-particle hopping operator spectrum is

$$\epsilon(q) := (\hat{t}_0 - \hat{t}_q) = \sum_{\alpha=1}^{d} 4\ \sin^2(q_\alpha/2) \geq 0, \quad q \in \Lambda^*, \tag{2.8}$$

with eigenfunctions $\{\hat{e}_q\}_{q\in\Lambda^*}$.

It is known that the lattice free Bose-gas (2.6) with *n.n.* hopping manifests the *zero-mode* BEC when $d > 2$, since the spectral density of states $\mathcal{N}_d(d\epsilon)$ corresponding to (2.7) is small enough to make the *critical* particle density $\rho_c^{\text{free}}(\beta)$ *bounded* for a given temperature β^{-1}:

$$\rho^{\text{free}}_{c,\ n.n.}(\beta) := \lim_{\mu\uparrow 0}\lim_{\Lambda} \frac{1}{V}\sum_{q\in\Lambda^*} \frac{1}{e^{\beta(\epsilon(q)-\mu)}-1} = \frac{1}{(2\pi)^d}\int_{\mathcal{B}^d} d^d q\ \frac{1}{e^{\beta\epsilon(q)}-1} \tag{2.9}$$

$$= \int_{\mathbb{R}_+} \mathcal{N}_d(d\epsilon)\ \frac{1}{e^{\beta\epsilon}-1} < \infty.$$

Here $\lim_\Lambda$ stands for the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$, by $\mathcal{B}^d := [-\pi, \pi]^d$ we denote the first *Brillouin zone* and the density of states $\mathcal{N}_d(d\epsilon) = \{c_d\epsilon^{(d/2-1)} + o(\epsilon^{(d/2-1)})\}d\epsilon$ for small ϵ.

A similar result is true for the *Infinite-Range* Hopping (IRH) Laplacian:

$$t^{\Lambda}_{x\,y} = \frac{1}{V}(1 - \delta_{x\,,\,y}), \quad x, y \in \Lambda. \tag{2.10}$$

By (2.10) the one-particle spectrum in this case takes the form:

$$\epsilon(q) := (\hat{t}_0 - \hat{t}_q) = (1 - \delta_{q,0}) \geq 0, \quad q \in \Lambda^*. \tag{2.11}$$

Therefore, it has a *gap*:

$$\lim_{q\to 0} \epsilon(q) = 1 \neq \epsilon(0) = 0, \tag{2.12}$$

and allowed values of the chemical potential are still $\mu \leq 0$. Since the density of states is simply zero in the gap, and $|\Lambda^*| = V|\mathcal{B}^d|$, we have $\mathcal{N}_d(d\epsilon) = \delta(\epsilon - 1)d\epsilon$. Therefore, the *critical* particle density has a bounded value:

$$\rho^{\text{free}}_{c,\ i.r.}(\beta) = \frac{1}{(2\pi)^d}\int_{\mathcal{B}^d} d^d q\ \frac{1}{e^{\beta}-1} = \frac{1}{e^{\beta}-1} < \infty, \tag{2.13}$$

for any dimensions. The latter implies a zero-mode BEC for densities $\rho > \rho_{c,\, i.r.}^{\mathrm{free}}(\beta)$.

The problem of existence of BEC gets much less obvious if one takes into account the *boson interaction*. This is even the case for the simplest *on-site* repulsive interaction

$$H_\Lambda := T_\Lambda + \lambda \sum_{x\in\Lambda} n_x(n_x - 1),\ \lambda \geq 0, \tag{2.14}$$

known as the *Bose–Hubbard* model. (Notice that *attraction*: $\lambda < 0$ makes this model unstable, see Ref. 2 for discussion of other cases.)

Remark 2.1. *Concerning the model (2.14) the best rigorous results so far are:*

- *a proof of BEC for the n.n. lattice Laplacian and the hard-core boson repulsion: $\lambda = +\infty$, by Ref. 14 for the case of the half-filled lattice, see also Ref. 15;*
- *a recent exact solution of the IRH Bose–Hubbard model (2.10), (2.14) for any $\lambda \geq 0$ by Ref. 4.*

The aim of the the present paper is to study a *disordered* IRH Bose–Hubbard model. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. We define our basic model by the random Hamiltonian:

$$H_\Lambda^\omega = \frac{1}{2V} \sum_{x,y\in\Lambda} (a_x^* - a_y^*)(a_x - a_y) + \sum_{x\in\Lambda} \lambda_x^\omega n_x(n_x - 1) + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x, \tag{2.15}$$

where parameters $\{\lambda_x^\omega \geq 0\}_{x\in\mathbb{Z}^d}$ and $\{\varepsilon_x^\omega \in \mathbb{R}^1\}_{x\in\mathbb{Z}^d}$, for $\omega \in \Omega$, are real-valued random fields on $\mathbb{Z}^d$.

Let $D \subset \mathbb{Z}^d$ be a finite subset of the lattice points $D := \{x_j\}_j$. For any $y \in \mathbb{Z}^d$ the set $\tau_y(D) = \{x_j + y\}_j$ is a translation of the subset D by the vector y. With a real-valued random field $\{\xi_x^\omega\}_{x\in\mathbb{Z}^d}$, $\omega \in \Omega$, one can associate the family of consistent *finite-dimensional distributions*:

$$P_{\xi,D}(\{B_x\}_{x\in D}) := \mathbb{P}\left\{\omega \in \Omega : \xi_x^\omega \in B_x, x \in D\right\},$$

corresponding to subsets $D \subset \mathbb{Z}^d$ and Borel sets $\{B_x\}_{x\in D}$. Then this random field is called *stationary* (or *homogeneous*) if these finite-dimensional distributions are translation-invariant:

$$P_{\xi,D}(\{B_x\}_{x\in D}) = P_{\xi,\tau_y(D)}(\{B_z\}_{z\in\tau_y(D)}), \quad y \in \mathbb{Z}^d.$$

If, in addition, one has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} P_{\xi,D_1 \cup \tau_y^k(D_2)} = P_{\xi,D_1}\, P_{\xi,D_2}$$

for any $y \in \mathbb{Z}^d$, $y \neq 0$, and finite subsets $D_{1,2} \subset \mathbb{Z}^d$, then the stationary random field $\{\xi_x^\omega\}_{x\in\mathbb{Z}^d}$ is called *ergodic*. A particular example of a stationary ergodic field is a field $\{\xi_x^\omega\}_{x\in\mathbb{Z}^d}$ of *independent identically distributed* (*i.i.d.*) random variables.

Below we assume that the random fields $\{\lambda_x^\omega \geq 0\}_{x\in\mathbb{Z}^d}$ and $\{\varepsilon_x^\omega \in \mathbb{R}^1\}_{x\in\mathbb{Z}^d}$, for $\omega \in \Omega$, are *stationary* and *ergodic*.

We denote by

$$p_\Lambda^\omega(\beta,\mu) := p\left[H_\Lambda^\omega\right](\beta,\mu) := \frac{1}{\beta V}\mathrm{Tr}_{\mathfrak{F}_B(\Lambda)} \exp\left\{-\beta\left(H_\Lambda^\omega - \mu N_\Lambda\right)\right\} \tag{2.16}$$

the grand canonical pressure of the system (2.15) for given temperature β^{-1} and chemical potential μ. For *non-random* parameters $\lambda_x^\omega = \lambda \geq 0$ and $\varepsilon_x^\omega = \varepsilon = 0$ the model (2.15) was considered in ref. 4.

Our main theorem is a formula for the pressure of this model given some general regularity conditions on the random parameters involved in the Hamiltonian (2.15).

Theorem 2.1. *Let the stationary, ergodic random fields $\{\lambda_x^\omega\}_{x\in\mathbb{Z}^d}$ and $\{\varepsilon_x^\omega\}_{x\in\mathbb{Z}^d}$ be such that:*

$$\lambda_{\min} := \inf_{x,\omega} \lambda_x^\omega > 0, \qquad \varepsilon_{\min} := \inf_{x,\omega} \varepsilon_x^\omega > -\infty. \tag{2.17}$$

Then for almost all $\omega \in \Omega$, i.e., almost sure (a.s.), there exists a non-random thermodynamic limit of the pressure (2.16):

$$a.s. - \lim_\Lambda p_\Lambda^\omega(\beta,\mu) = p(\beta,\mu), \tag{2.18}$$

such that

$$\begin{aligned} p(\beta,\mu) := \sup_{r\geq 0}\Big\{ -r^2 + \beta^{-1}\mathbb{E}\Big\{ \ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp\beta\Big[(\mu - \varepsilon_x^\omega - 1)n_x \\ - \lambda_x^\omega n_x(n_x - 1) + r(a_x^* + a_x)\Big]\Big\}\Big\}, \end{aligned} \tag{2.19}$$

where $\mathbb{E}(\cdot)$ is the expectation with respect to the measure $\mathbb{P}$. $(\mathfrak{F}_B)_x$ denotes the single-site Fock space, i.e. $(\mathfrak{F}_B)_x = \mathfrak{F}_B(\{x\})$.

Remark 2.2. *Note that conditions (2.17), and in particular the first one, ensure superstability of the Hamiltonian (2.15). The proof of (2.18) for $\lambda_{\min} = 0$ needs another technique than that for Theorem 2.1. In Section 3.2 we consider a particular case of non-random $\{\lambda_x^\omega = 0\}_{x\in\mathbb{Z}^d}$ (perfect bosons).*

We prove the main theorem in the next subsection.

2.2. Proof of the Main Theorem

We first prove the following

Lemma 2.1. *Let us add to Hamiltonian (2.15) the sources*[(16)]*:*

$$H_{\Lambda}^{\omega}(\nu) := H_{\Lambda}^{\omega} - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*), \quad \nu \in \mathbb{C}, \tag{2.20}$$

and define the corresponding approximating Hamiltonian:

$$H_{\Lambda}^{\omega}(z,\nu) := H_{0\Lambda}^{\omega}(z) - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*), \tag{2.21}$$

where

$$H_{0\Lambda}^{\omega}(z) := H_{0\Lambda}^{\omega} + V\,|z|^2 - \sqrt{V}(\overline{z}\hat{a}_0 + z\hat{a}_0^*), \quad z \in \mathbb{C}, \tag{2.22}$$

and

$$H_{0\Lambda}^{\omega} := \sum_{x\in\Lambda} \lambda_x^{\omega} n_x(n_x - 1) + \sum_{x\in\Lambda} \left(\varepsilon_x^{\omega} + 1\right) n_x. \tag{2.23}$$

Then for all $\omega \in \Omega$ and $\mu \in \mathbb{R}^1$ one obtains in the disk $|\nu| \le C_0$ the estimate

$$0 \le p\left[H_{\Lambda}^{\omega}(\nu)\right] - p\left[H_{\Lambda}^{\omega}(z_{\Lambda,\omega}(\nu),\nu)\right] \le \frac{1}{V}\left\{u + w\,\beta^{-1}\partial_{\nu}\,\partial_{\overline{\nu}}\, p\left[H_{\Lambda}^{\omega}(\nu)\right]\right\}. \tag{2.24}$$

for some constants $u > 0$, $w > 0$.

Proof: By definitions (2.5), (2.6), (2.11) and (2.23) the Hamiltonian (2.15) takes the form

$$H_{\Lambda}^{\omega} = T_{\Lambda} + \sum_{x\in\Lambda} \lambda_x^{\omega} n_x(n_x - 1) + \sum_{x\in\Lambda} \varepsilon_x^{\omega} n_x = -\hat{a}_0^*\hat{a}_0 + H_{0\Lambda}^{\omega}. \tag{2.25}$$

Since conditions (2.17) imply the estimate from below:

$$\begin{aligned} H_{\Lambda}^{\omega} &\ge -\hat{a}_0^*\hat{a}_0 + N_{\Lambda} + \lambda_{\min} \sum_{x\in\Lambda} n_x(n_x - 1) + \varepsilon_{\min} N_{\Lambda} \\ &\ge \frac{\lambda_{\min}}{V} N_{\Lambda}^2 + (\varepsilon_{\min} - \lambda_{\min}) N_{\Lambda}, \end{aligned} \tag{2.26}$$

the Hamiltonian (2.25) is *superstable*. Thus, the pressure in (2.18) is defined for *all* $\mu \in \mathbb{R}^1$.

By (2.20) and (2.21) we have

$$H_{\Lambda}^{\omega}(\nu) - H_{\Lambda}^{\omega}(z,\nu) = -(\hat{a}_0 - z\sqrt{V})^*(\hat{a}_0 - z\sqrt{V}), \tag{2.27}$$

and by virtue of the Bogoliubov *convexity inequality* one gets the estimates:

$$0 \le p\left[H_{\Lambda}^{\omega}(\nu)\right] - p\left[H_{\Lambda}^{\omega}(z,\nu)\right] \le \frac{1}{V}\left\langle(\hat{a}_0 - z\sqrt{V})^*(\hat{a}_0 - z\sqrt{V})\right\rangle_{H_{\Lambda}^{\omega}(\nu)} \tag{2.28}$$

for *each* realization $\omega \in \Omega$. Here $\langle - \rangle_{H_\Lambda^\omega(\nu)} := \langle - \rangle_{H_\Lambda^\omega(\nu)}(\beta, \mu)$ denotes the grand-canonical quantum Gibbs state with Hamiltonian (2.20), and from now on we systematically omit the arguments (β, μ). If we choose in the right-hand side of (2.28)

$$z = \frac{1}{\sqrt{V}} \langle \hat{a}_0 \rangle_{H_\Lambda^\omega(\nu)}, \tag{2.29}$$

then (2.28) implies the following estimate for each $\omega \in \Omega$:

$$0 \leq p\left[H_\Lambda^\omega(\nu)\right] - \sup_{z \in \mathbb{C}} p\left[H_\Lambda^\omega(z, \nu)\right] \leq \frac{1}{V} \left\langle \delta\hat{a}_0^* \, \delta\hat{a}_0 \right\rangle_{H_\Lambda^\omega(\nu)}, \tag{2.30}$$

where we denote

$$\delta\hat{a}_0 := \hat{a}_0 - \langle \hat{a}_0 \rangle_{H_\Lambda^\omega(\nu)}. \tag{2.31}$$

Since (2.5) implies the estimates:

$$-\sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*) \geq -|\nu|^2 \, \hat{a}_0^*\hat{a}_0 - V \geq -|\nu|^2 N_\Lambda - V, \tag{2.32}$$

by virtue of (2.26) and (2.32) the Hamiltonian with sources (2.20) is also *super-stable*:

$$H_\Lambda^\omega(\nu) \geq \frac{\lambda_{\min}}{V} N_\Lambda^2 + (\varepsilon_{\min} - \lambda_{\min} - |\nu|^2)\, N_\Lambda - V, \tag{2.33}$$

uniformly in $\omega \in \Omega$ and in $|\nu| \leq C_0$, for a fixed $C_0 \geq 0$. The superstability (2.33) implies that there is a *monotonous nondecreasing* function $M := M(\beta, \mu) \geq 0$ of $\mu \in \mathbb{R}^1$, such that for any $\omega \in \Omega$ we have the bounds:

$$\begin{aligned} \left|\left\langle \hat{a}_0/\sqrt{V} \right\rangle_{H_\Lambda^\omega(\nu)} (\beta, \mu)\right|^2 &= \left|\partial_{\overline{\nu}}\, p\left[H_\Lambda^\omega(\nu)\right](\beta, \mu)\right|^2 \\ &\leq \langle N_\Lambda / V \rangle_{H_\Lambda^\omega(\nu)} (\beta, \mu) = \partial_\mu \, p\left[H_\Lambda^\omega(\nu)\right](\beta, \mu) \leq M^2, \end{aligned} \tag{2.34}$$

and

$$\left|z_{\Lambda,\omega}(\beta, \mu; \nu)\right|^2 \leq M^2 \tag{2.35}$$

for the *maximizer* $z_{\Lambda,\omega}(\nu) := z_{\Lambda,\omega}(\beta, \mu; \nu)$ in (2.30):

$$p\left[H_\Lambda^\omega(z_{\Lambda,\omega}(\beta, \mu; \nu), \nu)\right](\beta, \mu) := \sup_{z \in \mathbb{C}} p\left[H_\Lambda^\omega(z, \nu)\right](\beta, \mu), \tag{2.36}$$

uniform in $|\nu| \leq C_0$. Notice that the maximizer satisfies the equation:

$$z_{\Lambda,\omega}(\nu) = \partial_{\overline{\nu}} p\left[H_\Lambda^\omega(z_{\Lambda,\omega}(\nu), \nu)\right] = \left\langle \hat{a}_0/\sqrt{V} \right\rangle_{H_\Lambda^\omega(z_{\Lambda,\omega}(\nu),\ \nu)}. \tag{2.37}$$

Moreover, by the same line of reasoning as in Ref. 20, Ch. 4 (see also Ref. 4) one gets that for $|\nu| < C_0$ there are some $u = u(M) > 0$ and $w = w(M) > 0$ such that

$$\left\langle \delta\hat{a}_0^* \, \delta\hat{a}_0 \right\rangle_{H_\Lambda^\omega(\nu)} \leq \left\{ u + w(\delta\hat{a}_0^* \, , \delta\hat{a}_0)_{H_\Lambda^\omega(\nu)} \right\} , \tag{2.38}$$

where

$$(\delta\hat{a}_0^* \, , \delta\hat{a}_0)_{H_\Lambda^\omega(\nu)} = \beta^{-1} \partial_\nu \, \partial_{\overline{\nu}} \, p \left[H_\Lambda^\omega(\nu) \right] . \tag{2.39}$$

Then the estimates (2.30), (2.38) and definition (2.39) imply the estimate (2.24). □

Proof of Theorem 2.1: Following[18] we define in the Hilbert space $L^2(\{(\mathrm{Re}\nu, \ \mathrm{Im}\nu) \in \mathbb{R}^2 : |\nu| < C_0\})$ the *Dirichlet* self-adjoint extension $\hat{L}_V$ of the operator

$$L_V := I - w(\beta V)^{-1} \, \partial_\nu \, \partial_{\overline{\nu}} \, . \tag{2.40}$$

Here $4\partial_\nu \, \partial_{\overline{\nu}} = \Delta$ coincides with the two-dimensional Laplacian operator in variables $(\mathrm{Re}\nu, \ \mathrm{Im}\nu)$. The operator $\hat{L}_V$ is invertible and $\hat{L}_V^{-1}$ has the kernel $(\hat{L}_V^{-1})(\nu, \nu')$ (*Green* function), and $(\hat{L}_V^{-1})(\nu, \nu') = 0$ for $|\nu| = C_0$, or $|\nu'| = C_0$, by the Dirichlet boundary condition. Since the semigroup $\{\exp{[-t(\hat{L}_V - I)]}\}_{t \geq 0}$ is *positivity preserving*, the same property is true for the operator $\hat{L}_V^{-1}$, see e.g., Ref. 19 Ch.X.4.

Now, let $p(\nu) := p[H_\Lambda^\omega(\nu)]$ and $p_0(\nu) := p[H_\Lambda^\omega(z_{\Lambda,\omega}(\nu), \nu)]$. Since $\hat{L}_V^{-1}$ is positivity preserving, then (2.24)–(2.40) imply

$$\left(\hat{L}_V^{-1}(p_0 + u/V) \right)(\nu) \geq p(\nu) , \tag{2.41}$$

and by consequence the estimates

$$\begin{aligned} 0 \leq p \left[H_\Lambda^\omega(\nu) \right] - p \left[H_\Lambda^\omega(z_{\Lambda,\omega}(\nu), \nu) \right] &\leq \left(\hat{L}_V^{-1}(p_0 + u/V) \right)(\nu) - p_0(\nu) \\ &\leq \int_{|\nu'| < C_0} d\nu' \left(\hat{L}_V^{-1} \right)(\nu, \nu') \left\{ p_0(\nu') - p_0(\nu) \right\} + u/V, \end{aligned} \tag{2.42}$$

where we used that $\int_{|\nu'|<C_0} d\nu' (\hat{L}_V^{-1})(\nu, \nu') = 1$, $|\nu| < C_0$. By virtue of (2.35) and (2.37) we obtain for the integral in the right-hand side of (2.42) the estimate:

$$\begin{aligned} &\int_{|\nu'| < C_0} d\nu' \left(\hat{L}_V^{-1} \right)(\nu, \nu') \left\{ p_0(\nu') - p_0(\nu) \right\} \leq \\ &\qquad 2M \int_{|\nu'| < C_0} d\nu' \left(\hat{L}_V^{-1} \right)(\nu, \nu') \left| \nu' - \nu \right| = I_V . \end{aligned} \tag{2.43}$$

After change of variables to $\xi = \nu\sqrt{V}$, we get

$$I_V = \frac{2M}{V}\int_{|\xi'|<C_0\sqrt{V}} d\xi' \left(\hat{L}^{-1}_{V=1}\right)(\xi,\xi')\left|\xi'-\xi\right| \leq \frac{\tilde{M}}{V}. \tag{2.44}$$

Here we used that in $\mathbb{R}^2$ the *Green* function is known explicitly:

$$\left(\hat{L}^{-1}_{\infty}\right)(\xi,\xi') = \frac{w}{2\pi\beta}K_0\left(\frac{\beta}{w}\left|\xi-\xi'\right|\right), \tag{2.45}$$

where the Bessel function $K_0(x) \simeq \sqrt{\pi/2x}\exp(-x)$ decays exponentially fast for large $x > 0$. Therefore, (2.42) and (2.44) imply

$$0 \leq p\left[H^{\omega}_{\Lambda}(\nu)\right] - p\left[H^{\omega}_{\Lambda}(z_{\Lambda,\omega}(\nu),\nu)\right] \leq O(1/V), \tag{2.46}$$

for all $\omega \in \Omega$, any $\beta > 0$, $\mu \in \mathbb{R}^1$ and $|\nu| < C_0$.

Notice that by definitions (2.23) and (2.22) for any $z,\nu \in \mathbb{C}$ we get:

$$p^{\omega}_{\Lambda,\,\mathrm{appr}}(\beta,\mu;z,\nu) := p\left[H^{\omega}_{\Lambda}(z,\nu)\right](\beta,\mu) = -|z|^2 + \frac{1}{\beta V}\sum_{x\in\Lambda}\ln \mathrm{Tr}_{\mathfrak{F}_x}$$
$$\times \exp\beta\left[\left(\mu-\varepsilon^{\omega}_x-1\right)n_x - \lambda^{\omega}_x n_x(n_x-1) + (z+\nu)a^*_x + (\overline{z}+\overline{\nu})a_x\right]. \tag{2.47}$$

Then ergodicity of the random fields $\left\{\lambda^{\omega}_x\right\}_{x\in\mathbb{Z}^d}$ and $\left\{\varepsilon^{\omega}_x\right\}_{x\in\mathbb{Z}^d}$ implies the existence of the *a.s.* limit:

$$p_{\,\mathrm{appr}}(\beta,\mu;z,\nu) = a.s. - \lim_{\Lambda} p^{\omega}_{\Lambda,\,\mathrm{appr}}(\beta,\mu;z,\nu) = -|z|^2 + \beta^{-1}\mathbb{E}$$
$$\left\{\ln \mathrm{Tr}_{\mathfrak{F}_x}\exp\beta\left[\left(\mu-\varepsilon^{\omega}_x-1\right)n_x - \lambda^{\omega}_x n_x(n_x-1) + (z+\nu)a^*_x + (\overline{z}+\overline{\nu})a_x\right]\right\}, \tag{2.48}$$

i.e., the *self-averaging*[17] of the limiting approximating pressure $p^{\omega}_{\mathrm{appr}}(\beta,\mu;z,\nu)$.

Now we put the source $\nu \to 0$ and we make the canonical (*gauge*) transformation:

$$\tilde{a}_x := a_x e^{i\,arg\,z}. \tag{2.49}$$

Since Hamiltonian (2.22) is invariant with respect of this transformation, we get that $z = |z| := r$ and (cf. (2.47)):

$$\tilde{p}^{\omega}_{\Lambda,\,\mathrm{appr}}(\beta,\mu;r) := p^{\omega}_{\Lambda,\,\mathrm{appr}}(\beta,\mu;z=r,\nu=0) = p\left[H^{\omega}_{\Lambda}(r,0)\right](\beta,\mu) = -r^2$$
$$+\frac{1}{\beta V}\sum_{x\in\Lambda}\ln \mathrm{Tr}_{\mathfrak{F}_x}\exp\beta\left[\left(\mu-\varepsilon^{\omega}_x-1\right)n_x - \lambda^{\omega}_x n_x(n_x-1) + r(\tilde{a}^*_x+\tilde{a}_x)\right]. \tag{2.50}$$

Therefore, without source the *maximizers* in (2.36) can be defined only up to a phase and their moduli satisfy the Equation:

$$r = \frac{1}{2V}\sum_{x\in\Lambda}\langle \tilde{a}_x + \tilde{a}_x^*\rangle_{H_\Lambda^\omega(r,0)} =: \xi_\Lambda^\omega(r), \tag{2.51}$$

where

$$\xi_x^\omega(r) := \langle \tilde{a}_x + \tilde{a}_x^*\rangle_{H_\Lambda^\omega(r,0)}$$
$$= \frac{\mathrm{Tr}_{\mathfrak{F}_x}\left\{(\tilde{a}_x + \tilde{a}_x^*)\exp\beta\left[(\mu - \varepsilon_x^\omega - 1)n_x - \lambda_x^\omega n_x(n_x - 1) + r(\tilde{a}_x^* + \tilde{a}_x)\right]\right\}}{\mathrm{Tr}_{\mathfrak{F}_x}\exp\beta\left[(\mu - \varepsilon_x^\omega - 1)n_x - \lambda_x^\omega n_x(n_x - 1) + r(\tilde{a}_x^* + \tilde{a}_x)\right]}. \tag{2.52}$$

When $r = 0$, the approximating Hamiltonian (2.22) is invariant with respect to canonical gauge transformations $\mathcal{U}_\varphi \tilde{a}_x \mathcal{U}_\varphi^* = \tilde{a}_x e^{i\varphi}$ for any φ. This implies $\xi_x^\omega(r = 0) = 0$. Hence, Eq. (2.51) always has a trivial solution $r = 0$ and , moreover, by (2.35) any nontrivial solution $r_\Lambda^\omega \le M$.

Finally, differentiating (2.52) with respect to r we obtain:

$$0 \le \partial_r \xi_x^\omega(r) \le R, \tag{2.53}$$

where, by the superstability (2.33), the upper bound R is finite uniformly in ω, r, x. Hence, $-2M \le \partial_r \tilde{p}_{\Lambda,\mathrm{appr}}^\omega(\beta,\mu;r) \le 2RM$ for $r \in [0, M]$. By consequence the limit (2.48) implies the *uniform a.s.* convergence of the sequence $\{\tilde{p}_{\Lambda,\mathrm{appr}}^\omega(\beta,\mu;r)\}_\Lambda$ for $r \in [0, M]$:

$$\tilde{p}_{\mathrm{appr}}(\beta,\mu;r) = a.s. - \lim_\Lambda \tilde{p}_{\Lambda,\mathrm{appr}}^\omega(\beta,\mu;r) = -r^2 + \beta^{-1}\mathbb{E}$$
$$\left\{\ln \mathrm{Tr}_{\mathfrak{F}_x}\exp\beta\left[\left(\mu - \varepsilon_x^\omega - 1\right)n_x - \lambda_x^\omega n_x(n_x - 1) + r(\tilde{a}_x^* + \tilde{a}_x)\right]\right\}, \tag{2.54}$$

Therefore,

$$a.s. - \lim_\Lambda \sup_{r\ge 0} \tilde{p}_{\Lambda,\mathrm{appr}}^\omega(\beta,\mu;r) = \sup_{r\ge 0} \tilde{p}_{\mathrm{appr}}(\beta,\mu;r). \tag{2.55}$$

Together with (2.46) and (2.48), the limit (2.55) proves the assertions (2.18) and (2.19) of the theorem. □

Remark 2.3. *The function $\xi_x^\omega(r)$ is increasing in r by virtue of (2.53). Moreover, it has also been suggested that for any $x \in \mathbb{Z}^d$ and $\omega \in \Omega$, the function $r \mapsto \xi_x^\omega(r)$ is concave, see Ref. 4 for discussion of this conjecture. This implies that the nontrivial solution of Eq. (2.51) is unique. Notice that homogeneity and ergodicity of the random field random field $\{\varepsilon_x^\omega\}_{x\in\mathbb{Z}^d}$ implies the same for the random field $\{\xi_x^\omega\}_{x\in\mathbb{Z}^d}$ defined by (2.52). Therefore, Eq. (2.51) in the thermodynamic limit takes* *the form:*

$$r = a.s. - \lim_{\Lambda} \xi_{\Lambda}^{\omega}(r) = \frac{1}{2}\, \mathbb{E}\left(\xi_{x=0}^{\omega}(r)\right) =: f(r), \tag{2.56}$$

expressing a self-averaging property of the order parameter r see Ref. 17. Since the expectation in (2.56) preserves convexity, the solution of the limiting Eq. (2.56) should also be unique. Therefore, with probability 1, the sequence of maximizers $\{r_{\Lambda}^{\omega}\}_{\Lambda}$ has a unique accumulation point in the interval $[0, M]$. Moreover, if r_{Λ}^{ω} is the unique solution of Eq. (2.51), then

$$a.s. - \lim_{\Lambda} r_{\Lambda}^{\omega} = r(\beta, \mu), \tag{2.57}$$

where $r(\beta, \mu)$ denotes the unique solution of Eq. (2.56).

To see this, note that since $\lambda_{\min} > 0$, by superstability we get $r_{\Lambda}^{\omega} \leq M$, see (2.35), i.e.

$$0 \leq \liminf_{\Lambda} r_{\Lambda}^{\omega} \leq \limsup_{\Lambda} r_{\Lambda}^{\omega} \leq M, \tag{2.58}$$

for any $\omega \in \Omega$. Now suppose that there exists $\Omega_{>}$ with $\mathbb{P}(\Omega_{>}) > 0$ and a subsequence $\left\{r_{\Lambda_n}^{\omega}\right\}_{n \geq 1}$, $\omega \in \Omega_{>}$ such that

$$\lim_{n \to \infty} r_{\Lambda_n}^{\omega} = r_{*}^{\omega} > r(\beta, \mu), \quad \omega \in \Omega_{>}. \tag{2.59}$$

Then, by virtue of (2.51), (2.53), (2.56) and (2.59) we get:

$$\xi_{\Lambda_n}^{\omega}(r_{*}^{\omega}) - R\left|r_{\Lambda_n}^{\omega} - r_{*}^{\omega}\right| \leq r_{\Lambda_n}^{\omega} = \xi_{\Lambda_n}^{\omega}(r_{*}^{\omega} + r_{\Lambda_n}^{\omega} - r_{*}^{\omega}) \leq \xi_{\Lambda_n}^{\omega}(r_{*}^{\omega}) + R\left|r_{\Lambda_n}^{\omega} - r_{*}^{\omega}\right|. \tag{2.60}$$

These estimates, together with the limit (2.59) and a.s.-convergence of $\xi_{\Lambda_n}^{\omega}(r)$ to $f(r)$ for any r imply

$$r_{*}^{\omega} = f(r_{*}^{\omega}) > r(\beta, \mu), \tag{2.61}$$

for any $\omega \in \Omega_{>}$ with $\mathbb{P}(\Omega_{>}) > 0$, which is impossible by uniqueness of solution of (2.56). Similarly one excludes the hypothesis $r_{*}^{\omega} < r(\beta, \mu)$, which proves (2.57).

3. LIMITING HAMILTONIANS

3.1. Limit of Hard-Core Bosons

The *hard-core* (h.c.) interaction in the Bose–Hubbard model (2.14) corresponds to $\lambda = +\infty$, or $\lambda_{\min} = +\infty$ for the IRH Bose–Hubbard model (2.15). This formally discards from the boson Fock space $\mathfrak{F}_B(\Lambda)$ all vectors with *more than one* particle at one site.

Let Φ_0 denote the *vacuum vector* in $\mathfrak{F}_B(\Lambda)$. Then the subspace $\mathfrak{F}_B^{h.c.}(\Lambda) \subset \mathfrak{F}_B(\Lambda)$, which corresponds to the hard-core restrictions, is spanned by the orthonormal vectors

$$\Phi_X = \prod_{x\in X} a_x^* \, \Phi_0, \quad X \subset \Lambda. \tag{3.1}$$

Since the subspace $\mathfrak{F}_B^{h.c.}(\Lambda)$ is closed, there is orthogonal projection P_Λ onto $\mathfrak{F}_B^{h.c.}(\Lambda)$ such that

$$\mathfrak{F}_B^{h.c.}(\Lambda) = P_\Lambda \, \mathfrak{F}_B(\Lambda), \tag{3.2}$$

and we get the representation

$$\mathfrak{F}_B(\Lambda) = \mathfrak{F}_B^{h.c.}(\Lambda) \oplus \left(\mathfrak{F}_B^{h.c.}(\Lambda)\right)^{\perp}, \tag{3.3}$$

where the orthogonal compliment $(\mathfrak{F}_B^{h.c.}(\Lambda))^{\perp} := (I - P)\mathfrak{F}_B(\Lambda)$.

Since our main Theorem 2.1 is valid for any $\lambda_{\min} > 0$ and the estimate (2.46) is uniform in λ_x^ω, we can extend this theorem to the hard-core case by taking the limit $\lambda_{\min} \to +\infty$.

For simplicity we consider the case of a sequence of non-random identical and increasing positive $\{\lambda_x^\omega = \lambda_s > 0\}_{s=1}^{\infty}$ such that $\lambda_s \to +\infty$.

Lemma 3.1. *Let $\lambda_s \to +\infty$. Then for all $\zeta \in \mathbb{C} : Im(\zeta) \neq 0$, and for any $\omega \in \Omega$ and $\nu \in \mathbb{C}$ we have the strong resolvent convergence of Hamiltonians (2.20):*

$$\lim_{\lambda_s\to+\infty} \left(H_\Lambda^\omega(s,\nu) - \zeta I\right)^{-1}\Psi$$
$$= P\left[T_\Lambda + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*) - \zeta I\right]^{-1} P\Psi, \quad \Psi \in \mathfrak{F}_B(\Lambda), \tag{3.4}$$

where

$$H_\Lambda^\omega(s,\nu) := T_\Lambda + \lambda_s \sum_{x\in\Lambda} n_x(n_x - 1) + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*). \tag{3.5}$$

The same is true for approximating Hamiltonians (2.21):

$$\lim_{\lambda_s\to+\infty} \left(H_\Lambda^{\omega,appr}(s,z,\nu) - \zeta I\right)^{-1}\Psi$$
$$= P\left[V\,|z|^2 - \sqrt{V}(\overline{z}\hat{a}_0 + z\hat{a}_0^*) + \sum_{x\in\Lambda}\left(\varepsilon_x^\omega + 1\right)n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*) - \zeta I\right]^{-1} P\Psi, \tag{3.6}$$

for any $z \in \mathbb{C}$ *and* $\Psi \in \mathfrak{F}_B(\Lambda)$. *Here*

$$H_\Lambda^{\omega,\mathrm{appr}}(s,z,\nu) := V\,|z|^2 - \sqrt{V}(\overline{z}\hat{a}_0 + z\hat{a}_0^*) + N_\Lambda + \lambda_s \sum_{x\in\Lambda} n_x(n_x - 1) + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*). \tag{3.7}$$

Proof: By estimate (2.33) and (3.5) for $0 < \lambda_s < \lambda_{s+1}$ we get:

$$\frac{\lambda_s}{V} N_\Lambda^2 + \left(\varepsilon_{\min} - \lambda_s - |\nu|^2\right) N_\Lambda - V \le H_\Lambda^\omega(s,\nu) \le H_\Lambda^\omega(s+1,\nu). \tag{3.8}$$

So, for any $\omega \in \Omega$ and $\nu \in \mathbb{C}$ Hamiltonians (3.5) form an increasing sequence of self-adjoint operators, semi-bounded from below. Let $\{h_s^\omega(\nu,\Lambda)[\Psi] := (\Psi, H_\Lambda^\omega(s,\nu)\Psi)_{\mathfrak{F}_B(\Lambda)}\}_{s=1}^\infty$ be the corresponding monotonic sequence of closed symmetric quadratic forms with domains $\operatorname{dom} h_s^\omega(\nu,\Lambda)$. Put

$$Q := \bigcap_{s\ge 1} \operatorname{dom} h_s^\omega(\nu,\Lambda), \tag{3.9}$$

and let $Q_0 = \overline{Q}$ be the closure of Q in the Hilbert space $\mathfrak{F}_B(\Lambda)$. Since for any $\omega \in \Omega$ and $\nu \in \mathbb{C}$

$$\lim_{\lambda_s\to+\infty} \left(\Psi, H_\Lambda^\omega(s,\nu)\Psi\right)_{\mathfrak{F}_B(\Lambda)} = +\infty, \quad \Psi \in \left(\mathfrak{F}_B^{h.c.}(\Lambda)\right)^\perp, \tag{3.10}$$

one gets $Q_0 = \mathfrak{F}_B^{h.c.}(\Lambda)$ and the strong resolvent convergence (3.4) of Hamiltonians, see e.g. Ref. 21, Ch. 4.4 or Ref. 22, Lemma 2.10. (Note that for hard cores the space $\mathfrak{F}_B^{h.c.}(\Lambda)$ is finite-dimensional, which makes these arguments even simpler.) The strong resolvent convergence (3.4) of Hamiltonians implies also

$$\lim_{\lambda_s\to+\infty} \left(\Phi, H_\Lambda^\omega(s,\nu)\Phi\right)_{\mathfrak{F}_B(\Lambda)} = (\Phi, P[T_\Lambda + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*)]P\Phi)_{\mathfrak{F}_B^{h.c.}(\Lambda)}, \quad \Phi \in \mathfrak{F}_B^{h.c.}(\Lambda). \tag{3.11}$$

The same line of reasoning leads to (3.6) for approximating Hamiltonians. □

By the Trotter approximating theorem[(23)] the convergence (3.4) and (3.6) yields the strong convergence of the Gibbs semigroups:

Corollary 3.1. *The following strong limits exist:*

$$\mathrm{s} - \lim_{\lambda_s\to+\infty} \mathrm{e}^{-\beta \mathrm{H}_\Lambda^\omega(\mathrm{s},\nu)} = \mathrm{e}^{-\beta \mathrm{H}_{\mathrm{h.c.},\Lambda}^\omega(\nu)}, \tag{3.12}$$

where

$$H_{h.c.,\Lambda}^\omega(\nu) := P[T_\Lambda + \sum_{x\in\Lambda} \varepsilon_x^\omega n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*)]P, \tag{3.13}$$

and similarly

$$\mathrm{s} - \lim_{\lambda_s \to +\infty} \mathrm{e}^{-\beta \mathrm{H}_\Lambda^{\omega,\mathrm{appr}}(\mathrm{s,z},\nu)} = \mathrm{e}^{-\beta \mathrm{H}_{\mathrm{h.c.},\Lambda}^{\omega,\mathrm{appr}}(\mathrm{z},\nu)}, \quad \mathrm{dom}\ \mathrm{H}_{\mathrm{h.c.},\Lambda}^{\omega}(\nu) = \mathfrak{F}_{\mathrm{B}}^{\mathrm{h.c.}}(\Lambda), \quad (3.14)$$

where

$$H_{h.c.,\Lambda}^{\omega,\mathrm{appr}}(z,\nu) := P[V\,|z|^2 - \sqrt{V}(\overline{z}\hat{a}_0 + z\hat{a}_0^*) + \sum_{x\in\Lambda}(\varepsilon_x^\omega + 1)n_x$$
$$-\sqrt{V}(\overline{\nu}\hat{a}_0 + \nu\hat{a}_0^*)]P\,, \quad (3.15)$$

with dom $H_{h.c.,\Lambda}^{\omega,appr}(z,\nu) = \mathfrak{F}_B^{h.c.}(\Lambda)$.

Since $\{e^{-\beta(H_\Lambda^\omega(s,\nu)-\mu N_\Lambda)}\}_{s\geq 1}$ is a sequence of trace-class operators from $\mathcal{C}_1(\mathfrak{F}_B(\Lambda))$ monotonously decreasing to the trace-class operator

$$e^{-\beta\left(H_{h.c.,\Lambda}^{\omega}(\nu)-\mu N_\Lambda\right)} \in \mathcal{C}_1(\mathfrak{F}_B^{h.c.}(\Lambda)),$$

the convergence (3.12) can be lifted to the trace-norm topology.[(24)] The same is true for (3.14). It then follows that the pressures also converge:

Lemma 3.2.

$$\lim_{\lambda_s\to+\infty} p\big[H_\Lambda^\omega(s,\nu)\big] = p\big[H_{h.c.,\Lambda}^\omega(\nu)\big], \quad (3.16)$$

$$\lim_{\lambda_s\to+\infty} p\big[H_\Lambda^\omega(s,z,\nu)\big] = p\big[H_{h.c.,\Lambda}^{\omega,appr}(z,\nu)\big]. \quad (3.17)$$

Since the estimate (2.46) is uniform in $\lambda \geq \lambda_{\min} > 0$, we can take the limit $\lambda_s \to +\infty$ to obtain

$$0 \leq p\left[H_{h.c.,\Lambda}^\omega(\nu)\right] - p\left[H_{h.c.,\Lambda}^{\omega,appr}(z_{\Lambda,\omega}(\nu),\nu)\right] \leq O(1/V), \quad (3.18)$$

for all $\omega \in \Omega$, any $\beta > 0$, $\mu \in \mathbb{R}^1$ and $|\nu| < C_0$. Then, by the same line of reasoning as after (2.46) in Theorem 2.1, we obtain the thermodynamic limit of the pressure for the hard-core bosons:

Corollary 3.2. *The pressure of the Infinite-Range-Hopping hard-core Bose–Hubbard model with randomness is given by*

$$p_{h.c.}(\beta,\mu) \quad (3.19)$$
$$= \sup_{r\geq 0}\left\{-r^2 + \beta^{-1}\mathbb{E}\big\{\ln Tr_{(\mathfrak{F}_B^{h.c.})_x} \exp(\beta P\left[\big(\mu - \varepsilon_x^\omega - 1\big)n_x + r(a_x^* + a_x)\right]P)\big\}\right\},$$

cf. expression (2.19) for finite λ.

Remark 3.1. *To calculate the trace over $\mathfrak{F}_B^{h.c.}$ note that the boson creation and annihilation operators are quite different from the operators : $c_x^* := P a_x^* P$, $c_x := P a_x P$ restricted to* dom $c_x^* =$ dom $c_x = \mathfrak{F}_B^{h.c.}$*, which occur in (3.19). The major difference consists in their commutation relations:*

$$[c_x, c_y^*] = 0, \quad (x \neq y), \qquad (c_x)^2 = (c_x^*)^2 = 0, \qquad c_x c_x^* + c_x^* c_x = I. \quad (3.20)$$

Taking the XY representation of relations (3.20) :

$$c_x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad c_x^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(3.19) gives the explicit formula

$$p_{h.c.}(\beta, \mu) = \sup_{r \geq 0} \left\{ -r^2 + \mathbb{E} \left\{ \frac{1}{2}\left(\mu - \varepsilon_x^\omega - 1\right) + \beta^{-1} \ln \left[2 \cosh \left(\frac{1}{2} \beta \sqrt{\left(\mu - \varepsilon_x^\omega - 1\right)^2 + 4r^2} \right) \right] \right\} \right\}, \quad (3.21)$$

for the grand-canonical pressure for the random IRH hard-core Bose–Hubbard model.

3.2. Limit of Perfect Bosons

The limit $\lambda \to 0$ is more delicate. For simplicity, below we assume that $\varepsilon_{\min} = 0$. Then Hamiltonian (2.15) for perfect bosons $\lambda_x^\omega = 0$ is non-negative, i.e. the corresponding pressure exists in a finite volume only for *negative* chemical potentials. There is an analogue of Lemma 3.1, if we subtract from this Hamiltonian a term μN_Λ with $\mu < 0$ and assume ν small enough:

Lemma 3.3. *Assume that $\varepsilon_{\min} = 0$ and let $\lambda_s \searrow 0$. Then for $\mu < 0$, for all $\zeta \in \mathbb{C} : Im(\zeta) \neq 0$, and for any $\omega \in \Omega$, we have the strong resolvent convergence of Hamiltonians (2.20):*

$$\lim_{\lambda_s \searrow 0} \left(H_\Lambda^\omega(s, \nu)\right) - \mu N_\Lambda - \zeta I)^{-1} \Psi \quad (3.22)$$

$$= \left\{ T_\Lambda + \sum_{x \in \Lambda} (\varepsilon_x^\omega - \mu) n_x - \sqrt{V}(\overline{\nu} \hat{a}_0 + \nu \hat{a}_0^*) - \zeta I \right\}^{-1} \Psi, \ \Psi \in \mathfrak{F}_B(\Lambda),$$

for $\nu \in \mathbb{C}$, if $|\nu|^2 < |\mu|$. The same is true for approximating Hamiltonians (2.21):

$$\lim_{\lambda_s \searrow 0} (H_\Lambda^{\omega, \mathrm{appr}}(s, z, \nu) - \mu N_\Lambda - \zeta I)^{-1} \Psi = \quad (3.23)$$

$$\left\{ V\, |z|^2 - \sqrt{V}\left(\overline{z} \hat{a}_0 + z \hat{a}_0^*\right) + \sum_{x \in \Lambda} (\varepsilon_x^\omega + 1 - \mu) n_x - \sqrt{V}(\overline{\nu} \hat{a}_0 + \nu \hat{a}_0^*) - \zeta I \right\}^{-1} \Psi,$$

for any $z \in \mathbb{C}$, $\zeta \in \mathbb{C} : Im(\zeta) \neq 0$ *and* $\Psi \in \mathfrak{F}_B(\Lambda)$.

Proof: The bound (2.33) now yields:

$$H_\Lambda^\omega(s, \nu, \mu) := H_\Lambda^\omega(s, \nu) - \mu N_\Lambda \geq (-\mu - |\nu|^2) N_\Lambda - V\,, \tag{3.24}$$

so that for $|\nu|^2 + \mu < 0$, the operators $\left\{H_\Lambda^\omega(s, \nu, \mu)\right\}_{s \geq 1}$ are positive. As in Lemma 3.1, for these operators we define the corresponding closed symmetric quadratic forms by $\{h_s^\omega(\nu, \mu, \Lambda)[\Psi] := (\Psi, H_\Lambda^\omega(s, \nu, \mu)\Psi)_{\mathfrak{F}_B(\Lambda)}\}_{s=1}^\infty$. Note that they are monotonously decreasing and bounded from below, which implies that for any $\omega \in \Omega$, $\nu \in \mathbb{C}$ and Λ the operators $\left\{H_\Lambda^\omega(s, \nu, \mu)\right\}_{s \geq 1}$ converge in the strong resolvent sense, see e.g., Ref. 25 Ch. VIII, to a positive self-adjoint operator $H_{\Lambda,0}^\omega(\nu, \mu)$. Let us define the symmetric form

$$h_\infty^\omega[\Phi] = \lim_{s \to \infty} h_s^\omega[\Phi], \tag{3.25}$$

with domain

$$\text{dom}\left(h_\infty^\omega\right) = \bigcup_{s \geq 1} \text{dom}\left(h_s^\omega\right).$$

It is known,[25] Ch. VIII, that if the form (3.25) is closable, then operator $H_{\Lambda,0}^\omega(\nu, \mu)$ is associated with the closure $\tilde{h}_\infty^\omega$. By explicit expression of $h_s^\omega(\nu, \mu, \Lambda)$ one gets that the limit form (3.25) is closable (and even closed), since it is associated with the self-adjoint operator $H_\Lambda^\omega(s = \infty, \nu, \mu)$. Then the operator $H_{\Lambda,0}^\omega(\nu, \mu)$ associated with the closure $\tilde{h}_\infty^\omega$ of (3.25) simply coincides with $H_\Lambda^\omega(s = \infty, \nu, \mu)$:

$$\tilde{h}_\infty^\omega[\Phi] = \left(\Phi\,,\ \left[T_\Lambda + \sum_{x \in \Lambda} \left(\varepsilon_x^\omega - \mu\right) n_x - \sqrt{V}(\overline{\nu}\hat{a}_0 + \nu \hat{a}_0^*)\right]\ \Phi\right),$$

that proves (3.22).

A similar argument applies for the approximating Hamiltonians (2.21). But, in contrast to the case of sources $|\nu|^2 < |\mu|$, that we can choose as small as we want to apply the main Theorem 2.1, the value of z will be defined by variational principle (2.19) with $\lambda_x^\omega \geq 0$. Now the semi-boundedness of $\left\{H_\Lambda^{\omega,appr}(s, z, \nu)\right\}_{s \geq 1}$ from below follows from the estimate

$$\sum_{x \in \Lambda} \left(\varepsilon_x^\omega + 1 - \mu\right) n_x - \sqrt{V}((\overline{\nu} + \overline{z})\hat{a}_0 + (\nu + z)\hat{a}_0^*) \geq -\ V\ \frac{|\nu + z|^2}{1 - \mu}. \tag{3.26}$$

The rest of the arguments is identical to those for the operators (3.24), or equivalently for the sequence $\{H_\Lambda^\omega(s, \nu)\}_{s \geq 1}$, and goes through verbatim to give the proof of the limit (3.23) with $H_\Lambda^{\omega,appr}(s = \infty, z, \nu) := H_{\Lambda,0}^{\omega,appr}(z, \nu)$. □

Corollary 3.3. *In a full analogy with Corollary 3.1 and Lemma 3.2, the Trotter approximation theorem and the monotonicity of the operator families*

$\{H_\Lambda^\omega(s,\nu)\}_{s\geq 1}$, $\{H_\Lambda^{\omega,\mathrm{appr}}(s,z,\nu)\}_{s\geq 1}$ *yield*

$$\lim_{\lambda_s\to 0} p\big[H_\Lambda^\omega(s,\nu)\big] = p\big[H_{\Lambda,0}^\omega(\nu)\big], \tag{3.27}$$

$$\lim_{\lambda_s\to 0} p\big[H_\Lambda^{\omega,\mathrm{appr}}(s,z,\nu)\big] = p\big[H_{\Lambda,0}^{\omega,\mathrm{appr}}(z,\nu)\big]. \tag{3.28}$$

Notice that, similarly to the Weakly Imperfect Bose-Gas, [20] *the estimate (2.46) for* $\mu < 0$ *is still uniform in* $\lambda \geq 0$*. Therefore, we can take there the limit* $\lambda_s \to 0$ *to obtain*

$$0 \leq p\left[H_{\Lambda,0}^\omega(\nu)\right] - p\left[H_{\Lambda,0}^{\omega,\mathrm{appr}}(z_{\Lambda,\omega}(\nu),\nu)\right] \leq O(1/V), \tag{3.29}$$

for all $\omega \in \Omega$*, any* $\beta > 0$ *and* $|\nu|^2 < -\mu$*. Then, following the same line of reasoning as after (2.46) in Theorem 2.1, we obtain the thermodynamic limit of the pressure for the perfect bosons:*

$$\begin{aligned} &p_0(\beta,\mu<0) \\ &= \sup_{r\geq 0}\left\{-r^2 + \beta^{-1}\mathbb{E}\big\{\ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp\left(\beta\left[\left(\mu-\varepsilon_x^\omega-1\right)n_x + r(a_x^*+a_x)\right]\right)\big\}\right\}, \end{aligned} \tag{3.30}$$

cf. expression (2.19) for finite λ*, where all values of* μ *are allowed. Since we put* $\varepsilon_{\min} = 0$*, the variational principle in (3.30) implies:*

$$\begin{aligned} p_0(\beta,\mu<0) &= \beta^{-1}\mathbb{E}\big\{\ln \mathrm{Tr}_{(\mathfrak{F}_B)_x} \exp\left(\beta\left[\left(\mu-\varepsilon_x^\omega-1\right)n_x\right]\right)\big\} \\ &= \beta^{-1}\mathbb{E}\left\{\ln\left[1-\exp\{\beta\left(\mu-\varepsilon_x^\omega-1\right)\}\right]^{-1}\right\}. \end{aligned} \tag{3.31}$$

The convexity of $\left\{p\left[H_{\Lambda,0}^\omega(\nu=0)\right]\right\}_\Lambda$ *and the thermodynamic limit* $p_0(\beta,\mu)$ *as the functions of* $\mu < 0$*, together with the Griffith lemma, see e.g. Ref. 20, yield the convergence of derivative with respect of* μ*, i.e. the formula for the total particle density:*

$$\rho(\beta,\mu<0) = \mathbb{E}\left[\frac{1}{e^{\beta(1+\varepsilon^\omega-\mu)}-1}\right]. \tag{3.32}$$

Remark 3.2. *As usual in the case of the perfect boson gas one recovers the value of thermodynamic parameters at extreme point* $\mu = 0$ *by continuation:* $\mu \to -0$*:*

$$p_0(\beta,\mu=0) := \beta^{-1}\mathbb{E}\left\{\ln\left[1-\exp\left\{\beta\left(-\varepsilon_x^\omega-1\right)\right\}\right]^{-1}\right\}, \tag{3.33}$$

$$\rho(\beta,\mu=0) := \mathbb{E}\left[\frac{1}{e^{\beta(1+\varepsilon^\omega)}-1}\right]. \tag{3.34}$$

In particular by (3.34) it gets clear that the gap (= 1) in the one-particle spectrum of the perfect boson gas T_Λ and $\varepsilon_{\min} = 0$ imply that the critical density

$$\rho_c(\beta) := \sup_{\mu<0} \rho(\beta, \mu) = \rho(\beta, \mu = 0) \tag{3.35}$$

is finite, cf. (2.12) and (2.13). This opens a room for the zero-mode Bose condensation in the case of the random potential $\{\varepsilon_x^\omega\}_x$.

4. PHASE DIAGRAM

Here we analyse only the case, when ε_x^ω is random, but the interaction couplings $\lambda_x^\omega = \lambda \geq 0$ are fixed.

To proceed we recall first the formulae determining the critical temperature $\beta_c(\rho, \lambda)^{-1}$ for the *nonrandom* case $\varepsilon_x^\omega = 0$. To this end we define, cf (2.50),

$$\tilde{p}(\beta, \mu, \lambda; r) := \frac{1}{\beta} \ln \mathrm{Tr}_{\mathcal{H}} \exp(-\beta\, [h_n(\mu, \lambda) - r(a^* + a)]), \tag{4.1}$$

where

$$h_n(\mu, \lambda) := (1 - \mu)n + \lambda n(n - 1). \tag{4.2}$$

The Hilbert space $\mathcal{H}$ stands for a typical $(\mathfrak{F}_B)_x$ and similarly, a and a^* stand for typical annihilation and creation operators a_x and a_x^* defined on $\mathcal{H}$. From Ref. 4 it is known that the critical temperature (and the critical chemical potential $\mu_c(\rho, \lambda)$) are defined, as functions of the total particle density ρ, by two equations:

$$\tilde{p}''(\beta, \mu, \lambda; 0) = 2, \quad \rho = \frac{1}{Z_0(\beta, \mu, \lambda)} \sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu,\lambda)}. \tag{4.3}$$

Here

$$\tilde{p}''(\beta, \mu, \lambda; 0) = \frac{2}{Z_0(\beta, \mu, \lambda)} \sum_{n=1}^{\infty} n \frac{e^{-\beta h_n(\mu,\lambda)} - e^{-\beta h_{n-1}(\mu,\lambda)}}{h_{n-1}(\mu, \lambda) - h_n(\mu, \lambda)}. \tag{4.4}$$

and

$$Z_0(\beta, \mu, \lambda) = \sum_{n=0}^{\infty} e^{-\beta h_n(\mu,\lambda)}.$$

If $\varepsilon_x^\omega \neq 0$ and $\lambda > 0$ then, by the Main Theorem 2.1 (see (2.19), (2.54) and (4.2)), to obtain the equations for the critical temperature and the critical chemical potential, we have to replace μ in (4.3) by $\mu - \varepsilon_x^\omega$ and average over ε_x^ω. This gives, instead of (4.3), the (*gap*) equation:

$$\mathbb{E}\left[\tilde{p}''(\beta, \mu - \varepsilon^\omega, \lambda; 0)\right] = 2, \tag{4.5}$$

and the equation for the density:

$$\rho = \mathbb{E}\left[\frac{1}{Z_0(\beta, \mu - \varepsilon^\omega, \lambda)} \sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu - \varepsilon^\omega,\, \lambda)}\right]. \tag{4.6}$$

The case of $\lambda = 0$ is more subtle, and we begin with it the next subsection.

4.1. Perfect Bosons: $\lambda = 0$

Without loss of generality, we can assume that the random ε^ω takes values in the interval $[0, \varepsilon]$. In that case the maximal allowed value for μ (i.e. the *critical value*) is still $\mu_c = 0$, and the critical inverse temperature $\beta_c := \beta_c(\rho, \lambda = 0)$ is given (see (3.34), (3.35)) by:

$$\rho = \mathbb{E}\left[\frac{1}{e^{\beta_c(1+\varepsilon^\omega)} - 1}\right]. \tag{4.7}$$

Remark that, *irrespective* of the distribution of ε^ω, the Eq. (4.7) implies that the resulting β_c is *lower* than $\ln(1 + \frac{1}{\rho})$, which corresponds to the nonrandom case $\varepsilon_x^\omega = 0$, i.e. *disorder enhances* Bose–Einstein condensation. We shall see (Sec. 4.3.3) that this is *no longer true* when $\lambda > 0$, and even that the *opposite* holds, if λ is small enough!

Notice that formula (4.7) is in agreement with the general expression found in Ref. 9:

$$\rho = \int \frac{d\bar{\mathcal{N}}(E)}{e^{\beta_c E} - 1}, \tag{4.8}$$

where $\bar{\mathcal{N}}(E)$ is the *integrated* density of states given by

$$\bar{\mathcal{N}}(E) = \text{a.s.}-\lim_{V\to\infty} \frac{1}{V}\#\big\{i : E_i^\omega \le E\big\}. \tag{4.9}$$

Here $\{E_i^\omega\}_{i\ge1}$ are the eigenvalues of the one-particle Hamiltonian with a random potential $\{\varepsilon_x^\omega\}_{x\in\Lambda}$:

$$\big(h_\Lambda^\omega u\big)(x) := (t_\Lambda u)(x) + \sum_{x\in\Lambda} \varepsilon_x^\omega u(x),\ \ x \in \Lambda,\ \ u\mathfrak{h}(\Lambda), \tag{4.10}$$

for *IR* kinetic-energy hopping, see (2.1), (2.10), and $\#\{i : E_i^\omega \le E\}$ counting the number of the corresponding eigenfunctions (including the *multiplicity*). It is known that for any *ergodic* random potential $\{\varepsilon_x^\omega\}_{x\in\Lambda}$, the limit (4.9) exists *almost surely* (a.s.) and that it is *non-random*, see e.g. Ref. 17. A contact between formulae (4.7) and (4.8) is given by the following

Lemma 4.1. *In the case of an i.i.d. random potential, the integrated density of states is equal to*

$$\bar{\mathcal{N}}(E) = \mathbb{P}\,[\varepsilon^{\omega} \leq E - 1] = \mathbb{E}\,[\theta(E - 1 - \varepsilon^{\omega})]. \tag{4.11}$$

Proof: Fix E and let $p = \mathbb{P}\,[\varepsilon^{\omega} \leq E - 1]$. By the Central Limit Theorem, for given $\delta > 0$, there exists $c > 0$ such that with probability $Pr > 1 - \delta$ the number of sites $x \in \Lambda$ with $\varepsilon_x^{\omega} \leq E - 1$ is in the interval $(pV - c\sqrt{V}, pV + c\sqrt{V})$. Given a configuration for which this is the case, let $\Lambda_\varepsilon \subset \Lambda$ be the set where $\varepsilon_x^{\omega} \leq E - 1$. Consider the states $\phi \in \mathfrak{h}(\Lambda)$ such that $\phi(x) = 0$, if $x \notin \Lambda_\varepsilon$ and $\sum_{x\in\Lambda} \phi(x) = 0$. Then

$$\left(h_{\Lambda}^{\omega}\phi\right)(x) = \frac{1}{V}\sum_{y=1}^{V}(\phi(x) - \phi(y)) + \varepsilon_x^{\omega}\phi(x) \leq E\phi(x), \quad x \in \Lambda_\varepsilon.$$

The space of such states ϕ has dimension $|\Lambda_\varepsilon| - 1$, so that

$$\#\left\{E_i^{\omega} \leq E\right\} \geq (|\Lambda_\varepsilon| - 1).$$

Dividing by V we get, in the limit $V \to \infty$,

$$\bar{\mathcal{N}}(E) \geq p.$$

Similarly, considering the eigenfunctions with supports concentrated on $\Lambda_\varepsilon^c = \Lambda \setminus \Lambda_\varepsilon$ we obtain

$$1 - \bar{\mathcal{N}}(E) \geq 1 - p.$$

Together, these estimates prove (4.11). □

The relations (4.11) show that the formulae (4.7) and (4.8) are equivalent. For details of a general statement see e.g. Ref. 17 Ch. II.5.

4.2. Discrete Random Potential and $\lambda > 0$

We now consider the case with interaction $\lambda > 0$, and first assume that the probability distribution of *i.i.d.* ε_x^{ω} is *discrete*.

A particularly simple case corresponds to the *hard-core* boson limit $\lambda = +\infty$, see Sec. 3. Then by (3.21) the equations for the critical value of the inverse temperature $\beta_c := \beta_c(\rho) = \beta_c(\rho, \lambda = +\infty)$ for a given density ρ, reduce to the system:

$$\mathbb{E}\left[\frac{\tanh\beta(\mu - \varepsilon^{\omega} - 1)/2}{\mu - \varepsilon^{\omega} - 1}\right] = 1 \tag{4.12}$$

and

$$\rho = \frac{1}{2} + \frac{1}{2}\mathbb{E}\left[\tanh \frac{1}{2}\beta(\mu - \varepsilon^{\omega} - 1)\right]. \tag{4.13}$$

The last Eq. (4.13) implies that for the hard-core interaction the total particle density has the estimate: $\rho \leq 1$.

4.2.1. Bernoulli Random Potential in the Hard-Core Limit $\lambda = +\infty$.

A special case of a discrete distribution is the Bernoulli distribution, where $\varepsilon_x^{\omega} = \varepsilon$ with probability p and $\varepsilon_x^{\omega} = 0$ with probability $1 - p$. We first consider the case $\lambda = +\infty$. The Eqs. (4.12) and (4.13) then read,

$$F_{p,\varepsilon}(\beta = \beta_c, \mu) := p\frac{\tanh \frac{1}{2}\beta_c(\mu - \varepsilon - 1)}{\mu - \varepsilon - 1} + (1 - p)\frac{\tanh \frac{1}{2}\beta_c(\mu - 1)}{\mu - 1} = 1 \tag{4.14}$$

and

$$G_{p,\varepsilon}(\beta = \beta_c, \mu) := \frac{1}{2} + \frac{1}{2}\left[p \tanh \frac{1}{2}\beta_c(\mu - \varepsilon - 1) + (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1)\right] = \rho. \tag{4.15}$$

Here a *new phenomenon* occurs for density $\rho = 1 - p$. To see this, we consider first a particular case of $p = 1/2$. Then $\rho = 1/2$, and by (4.15) we obtain, that the only possible solution for the corresponding chemical potential is $\mu(\rho = 1/2) := \mu(\rho = 1/2, \lambda = +\infty) = 1 + \varepsilon/2$. Inserting this value of μ into (4.14) we get for the critical temperature:

$$\tanh \frac{\beta_c \varepsilon}{4} = \frac{1}{2}\varepsilon.$$

This equation obviously has *no solution* for $\varepsilon \geq 2$. Therefore, there is *no* Bose–Einstein condensation for Bernoulli random potential, if $p = \rho = 1/2$, and ε is greater than some *critical* value: $\varepsilon_{cr} = 2$.

One can check that the same phenomenon occurs for $p \neq 1/2$ and for densities $\rho = 1 - p$, if ε is *large* enough, but now the reasoning is more delicate. First of all, by (4.14) and $\tanh u \leq u$ we see that in any case there is a *lower bound* on the inverse critical temperature:

$$\beta_c \geq 2. \tag{4.16}$$

Now assume that $p < 1/2$, i.e. $\rho > 1/2$. From (4.15) it then follows that for any ε one has

$$0 < \mu - 1 - \frac{1}{2}\varepsilon. \tag{4.17}$$

Indeed, if we suppose that $0 \leq \mu - 1 \leq \varepsilon/2$, then $\tanh \frac{1}{2}\beta_c(\mu - 1) \leq \tanh \frac{1}{2}\beta_c(1 + \varepsilon - \mu)$ and hence, by (4.15), we get

$$\begin{aligned} 2\rho - 1 &= p \tanh \frac{1}{2}\beta_c(\mu - \varepsilon - 1) + (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1) \\ &\leq (1 - 2p) \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) < 1 - 2p, \end{aligned}$$

contradicting our assumption $\rho = 1 - p$, if β_c exists and is finite.

Now notice that (4.15) with $\rho = 1 - p$ is equivalent to

$$\frac{1 - \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu)}{1 - \tanh \frac{1}{2}\beta_c(\mu - 1)} = \frac{1 - p}{p}. \tag{4.18}$$

The left-hand side of (4.18) can be estimated from below as

$$\frac{1 - \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu)}{1 - \tanh \frac{1}{2}\beta_c(\mu - 1)} = \frac{e^{\beta_c(\mu - 1 - \varepsilon/2)} + e^{-\beta_c \varepsilon/2}}{e^{-\beta_c(\mu - 1 - \varepsilon/2)} + e^{-\beta_c \varepsilon/2}} > e^{\beta_c(\mu - 1 - \varepsilon/2)}.$$

Together with (4.16) this yield an upper bound for (4.17):

$$0 < \mu - 1 - \frac{1}{2}\varepsilon < \frac{1}{\beta_c} \ln \frac{1 - p}{p} \leq \frac{1}{2} \ln \frac{1 - p}{p} < \frac{1 - 2p}{2p}. \tag{4.19}$$

But (4.19) implies that (4.14) has *no* solution β_c, since for *large* ε we obtain

$$\begin{aligned} & p \frac{\tanh \frac{1}{2}\beta_c(\mu - \varepsilon - 1)}{\mu - \varepsilon - 1} + (1 - p) \frac{\tanh \frac{1}{2}\beta_c(\mu - 1)}{\mu - 1} \\ & < \frac{p}{\varepsilon + 1 - \mu} + \frac{1 - p}{\mu - 1} < \frac{p}{\varepsilon/2 - (1 - 2p)/2p} + \frac{1 - p}{\varepsilon/2} < 1. \end{aligned} \tag{4.20}$$

We assumed that $p < 1/2$. Therefore by (4.20), our conclusion is true, in fact, for

$$\varepsilon \geq 1/p \geq 2 = \varepsilon_{cr}. \tag{4.21}$$

The same result follows in the case $p \geq 1/2$, if we interchange p and $1 - p$ and $\mu - 1$ and $1 + \varepsilon - \mu$ in the above argument.

Next we show that for any *other* $\rho \in (0, 1)$, i.e. for any $\rho \neq 1 - p$, the critical $\beta_c(\rho) < +\infty$, i.e. for these densities one always has Bose–Einstein condensation at low temperatures.

To this end suppose that there is $\rho^* \in (0, 1)$ such that $\rho^* \neq 1 - p$, but $\lim_{\rho \to \rho^*} \beta_c(\rho) = +\infty$. Then the left-hand side of (4.14) converges to

$$\lim_{\beta \to \infty} F_{p,\varepsilon}(\beta, \mu) = M_p(\mu, \varepsilon) := \frac{p}{|\mu - \varepsilon - 1|} + \frac{1 - p}{|\mu - 1|}. \tag{4.22}$$

The number of solutions of Eq. (4.14) in the limit $\lim_{\rho \to \rho^*} \beta_c(\rho) = +\infty$ depends on the value of $\varepsilon > 0$, but two singular points $\mu = 1$ and $\mu = 1 + \varepsilon$ of the function

(4.22) ensure (for nontrivial values of the probability: $p \neq 0$ and $p \neq 1$) that there are always at least *two solutions*: $\mu_1(\varepsilon) < 1$ and $\mu_2(\varepsilon) > 1 + \varepsilon$ of equation

$$M_p(\mu, \varepsilon) = 1. \tag{4.23}$$

If $\lim_{\rho \to \rho^*} \beta_c(\rho) = +\infty$, then for these two cases the Eq. (4.15) implies:

$$\rho^* = \lim_{\rho \to \rho^*} G_{p,\varepsilon}(\beta_c(\rho), \mu_1(\varepsilon) = 0,$$

$$\rho^* = \lim_{\rho \to \rho^*} G_{p,\varepsilon}(\beta_c(\rho), \mu_2(\varepsilon) = 1.$$

This contradicts our assumptions on ρ^* and makes impossible the hypothesis $\lim_{\rho \to \rho^*} \beta_c(\rho) = +\infty$.

Notice that the function $M_p(\mu, \varepsilon)$ has a *minimum* $\overline{\mu}(\varepsilon) \in (1, 1 + \varepsilon)$. If $M_p(\overline{\mu}(\varepsilon), \varepsilon) < 1$ (which is equivalent to $\varepsilon > \varepsilon_p := 1 + 2\sqrt{p(1-p)}$), then Eq. (4.23) has *two* complementary solutions $\mu_{\mp}(\varepsilon)$:

$$\mu_{\mp}(\varepsilon) = \frac{\varepsilon + 3}{2} - p \mp \sqrt{\left(\frac{\varepsilon - 1}{2}\right)^2 - p(1-p)}, \tag{4.24}$$

such that

$$1 < \mu_-(\varepsilon) < \overline{\mu}(\varepsilon) < \mu_+(\varepsilon) < 1 + \varepsilon.$$

If $\lim_{\rho \to \rho^*} \beta_c(\rho) = +\infty$, then for these two solutions Eq. (4.15) implies:

$$\rho^* = \lim_{\rho \to \rho^*} G_{p,\varepsilon}(\beta_c(\rho), \mu_{\mp}(\varepsilon)) = 1 - p,$$

This again contradicts our assumption about ρ^*, and thus proves the assertion: $\beta_c(\rho) < +\infty$ for any $\rho \neq 1 - p$.

Notice that by (4.24) the equation $M_p(\overline{\mu}(\varepsilon), \varepsilon) = 1$ has a unique solution $\varepsilon = \varepsilon_p \leq \varepsilon_{cr} = 2$, and one obtains $M_p(\overline{\mu}(\varepsilon), \varepsilon) > 1$ for all $\varepsilon < \varepsilon_p$, which excludes complementary solutions $\mu_{\mp}(\varepsilon)$. On the other hand, if

$$\varepsilon > \varepsilon_{cr} = \max_p \varepsilon_p = \varepsilon_{p=1/2}, \tag{4.25}$$

there are *always* complementary solutions (4.24). This may *restrict* the values of ρ, for which we have bounded critical $\beta_c(\rho)$, to a certain domain of densities.

To this end we consider first the ρ-independent Eq. (4.14). Notice that $F_{p,\varepsilon}(\beta, \mu)$ is a monotonously increasing function of β, so there is a *unique* solution $\tilde{\beta}_c(\mu)$ of equation (4.14) for a given μ, *if* there is one.

Since $(\tanh u)/u \leq 1$, then the left-hand side of (4.14) is *less* than 1, for $\beta \leq 2$. On the other hand, as $\beta \to \infty$, the left-hand side of (4.14) converges to $M_p(\mu, \varepsilon)$. Since the function (4.22) is singular at $\mu = 1$ and $\mu = 1 + \varepsilon$, a solution $2 < \tilde{\beta}_c(\mu) < +\infty$ for a certain μ always exists, and the set of those μ is defined by the condition:

$$S_{p,\varepsilon} := \left\{\mu \in \mathbb{R}^1 : \lim_{\beta\to\infty} F_{p,\varepsilon}(\beta,\mu) = M_p(\mu,\varepsilon) \geq 1\right\} \tag{4.26}$$

By (4.22) the set (4.26) for $\varepsilon > 0$ is a *compact* in $\mathbb{R}^1_+$. If there are no complementary solutions $\mu_{\mp}(\varepsilon)$, this compact is *connected*, but if

$$\varepsilon > \varepsilon_{cr}. \tag{4.27}$$

it contains two domains separated by a *gap*:

$$I(\varepsilon, p) := (\mu_-(\varepsilon)\,,\ \mu_+(\varepsilon)),$$

see (4.24). The gap $I(\varepsilon, p) \subset (1, 1+\varepsilon)$. There is no solutions $\tilde{\beta}_c(\mu)$ for $\mu \in I(\varepsilon, p)$ and for

$$\mu < (\varepsilon+1)/2 - \sqrt{((\varepsilon-1)/2)^2 - \varepsilon(1-p)},$$

or for

$$\mu > (\varepsilon+3)/2 + \sqrt{((\varepsilon+1)/2)^2 - \varepsilon(1-p)}.$$

Hence, for large ε (4.27) the set $S_{p,\varepsilon}$ is a union of two (separated by the gap $I(\varepsilon, p)$) bounded domains, which are vicinities of singular points $\mu = 1$ and $\mu = 1+\varepsilon$.

To understand, how the gap in the chemical potential for solution $\tilde{\beta}_c(\mu)$ modify the behaviour of $\beta_c(\rho)$, we have to consider the ρ -dependent Eq. (4.15). Notice that from (4.15) one obtains $\hat{\beta}_c(\mu,\rho)$ as a function of two variables. Therefore, $\beta_c(\rho)$ is a solution of equation:

$$\tilde{\beta}_c(\mu) = \hat{\beta}_c(\mu,\rho), \tag{4.28}$$

which in fact connects μ and ρ: $\overline{\mu}(\rho)$, i.e. $\beta_c(\rho) = \tilde{\beta}_c(\overline{\mu}(\rho)) = \hat{\beta}_c(\overline{\mu}(\rho), \rho)$.

Clearly, the left-hand side $G_{p,\varepsilon}(\beta,\mu)$ is increasing in μ and it tends to 0 as $\mu \to -\infty$ and to 1 as $\mu \to +\infty$. Excluding $\rho = 0$ or 1, there is therefore a *unique* solution $\mu(\beta,\rho)$ of (4.15) for each value of β. As $\beta \to 0$, $G_{p,\varepsilon}(\beta,\mu)$ tends to $1/2$ at constant μ. Therefore, if $\rho \neq 1/2$

$$\lim_{\beta\to 0} \mu(\beta,\rho) = \pm\infty,$$

depending on whether $\rho > 1/2$ or $\rho < 1/2$.

On the other hand, in the limit $\beta \to \infty$, we have that $G_{p,\varepsilon}(\beta,\mu)$: (a) tends to 0, if $\mu < 1$; (b) to $(1-p)/2$, if $\mu = 1$; (c) to $1-p$, if $1 < \mu < 1+\varepsilon$; (d) to $1-p/2$, if $\mu = 1+\varepsilon$, and (e) to 1, if $\mu > 1+\varepsilon$.

The $(a)-(e)$ give relation between ρ and μ for large β: if $0 < \rho < 1-p$, we must have $\mu(\beta,\rho) \to 1$ and, if $1-p < \rho < 1$, we obtain $\mu(\beta,\rho) \to 1+\varepsilon$, for $\beta \to \infty$. At $\rho = 1 - p$, we have to use the representation (4.18), that yields

$$\mu(\beta, \rho = 1 - p) = 1 + \frac{1}{2}\varepsilon - \frac{1}{2\beta} \ln \frac{p}{1-p} + o(\beta^{-1}), \tag{4.29}$$

if β is large. In particular, this justifies the remark (4.21) above about $\varepsilon_{\rm cr} = 2$, since $1 + \varepsilon/2$ lies in the gap $I(\varepsilon, p)$ only if $\varepsilon \geq 2 = \varepsilon_{\rm cr}$, see (4.24).

Hence, it follows that for $\rho \neq 1 - p$ two functions of μ corresponding to solutions (4.28) of Eqs. (4.14), (4.15) must intersect. On the other hand, (4.21) proves that they can not intersect for $\rho = 1 - p$, if $\varepsilon > \varepsilon_{\rm cr}$. In fact, we can derive *upper* bounds for $\beta_c(\rho)$ in the case $\rho \neq 1 - p$ and $|\rho - 1 + p|$ small.

To this end *we first consider the case* $\rho > 1 - p$. Let us assume $p \leq 1/2$. (The case $p > 1/2$ can be studied similarly.) Writing $\rho = 1 - p + \delta/2$ we present the Eq. (4.15) in the form

$$p \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) = (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1) + 2p - 1 - \delta. \tag{4.30}$$

Identity (4.30) implies that $\mu > 1 + \varepsilon/2$, since otherwise we get a contradiction:

$$1 - 2p + \delta = -p \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) + (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1) \leq$$

$$- p \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) + (1 - p) \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) \leq 1 - 2p.$$

On the other hand, for $\varepsilon \geq 1$, one gets the upper limit $\mu < \varepsilon + 1$. Indeed, if we suppose the opposite: $\mu \geq \varepsilon + 1$, then (4.15) and the general fact that $\beta_c \geq 2$ (see (4.16)) yield

$$1 - 2p + \delta = p \tanh \frac{1}{2}\beta_c(\mu - \varepsilon - 1) + (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1)$$

$$\geq (1 - p) \tanh \frac{1}{2}\beta_c(\mu - 1) \geq (1 - p) \tanh \varepsilon.$$

But this is impossible for (large) ε verifying:

$$\varepsilon > \frac{1}{2} \ln \frac{2 - 3p + \delta}{p - \delta}. \tag{4.31}$$

Therefore, we obtain for μ the *lower* and *upper* bounds:

$$1 + \varepsilon/2 < \mu < 1 + \varepsilon. \tag{4.32}$$

Now identity (4.30), together with the bounds (4.32), inequality $\tanh(u) > 1 - 2e^{-2u}$ and $\beta_c \geq 2$ (see (4.16)), yields the estimates:

$$1 - \frac{\delta}{p} - \frac{2}{p}e^{-\varepsilon} < \tanh \frac{1}{2}\beta_c(\varepsilon + 1 - \mu) < 1 - \frac{\delta}{p}. \tag{4.33}$$

$$1 > \frac{p - \delta - 2e^{-\varepsilon}}{\varepsilon + 1 - \mu} + (1 - p)\frac{1 - 2e^{-\varepsilon}}{\mu - 1} > \frac{\beta_c(p - \delta - 2e^{-\varepsilon})}{\ln(2p/\delta)}$$

and hence,

$$\beta_c < \frac{1}{p - \delta - 2e^{-\varepsilon}} \ln(2p/\delta). \tag{4.34}$$

The upper bound (4.34) holds for example, if $\delta < p/2$ and $\varepsilon > \ln(4/p)$.

Now *we consider the case* $\rho < 1 - p$ and suppose $p \leq 1/2$, since $p > 1/2$ can be studied similarly. Then we write: $\rho = 1 - p - \delta/2$. Eq. (4.15) now reads as

$$(1-p)\tanh\frac{1}{2}\beta_c(\mu - 1) = p\tanh\frac{1}{2}\beta_c(1 + \varepsilon - \mu) + 1 - 2p - \delta. \tag{4.35}$$

An argument similar to the case $\rho > 1 - p$ shows that

$$1 < \mu < 1 + \varepsilon, \tag{4.36}$$

if ε is large enough and $\delta < 1 - p$. Indeed, if we suppose the opposite: $\mu \geq 1 + \varepsilon$, then

$$1 - 2p - \delta \geq (1-p)\tanh\frac{1}{2}\beta_c(\mu - 1) \geq (1-p)\tanh\varepsilon,$$

which is impossible for

$$\varepsilon > \frac{1}{2}\ln\frac{2 - 3p - \delta}{p + \delta}.$$

Similarly, if we suppose that $\mu \leq 1$, then (4.35) implies

$$0 > p\tanh\frac{1}{2}\beta_c(1 + \varepsilon - \mu) + 1 - 2p - \delta > p\tanh\varepsilon \; + \; (1 - 2p - \delta),$$

which is impossible if $\delta < 1 - 2p$, or if $1 - 2p \leq \delta < 1 - p$ and

$$\varepsilon > \frac{1}{2}\ln\frac{3p - 1 + \delta}{1 - p - \delta}.$$

Now, (4.35) and (4.36) imply that

$$\tanh\frac{1}{2}\beta_c(\mu - 1) < 1 - \frac{\delta}{1-p}. \tag{4.37}$$

In the case $\mu \geq 1 + \frac{1}{2}\varepsilon$ this yields immediately the upper bound:

$$\beta_c < \frac{2}{\varepsilon}\ln\frac{2(1-p)}{\delta}. \tag{4.38}$$

On the other hand, if $1 < \mu < 1 + \varepsilon/2$, then by (4.35) and $\beta_c \geq 2$ we obtain

$$\begin{aligned}(1-p)\tanh\frac{1}{2}\beta_c(\mu - 1) &> p\tanh\frac{1}{4}\beta_c\varepsilon + 1 - 2p - \delta > p\tanh\frac{1}{2}\varepsilon \\ &+ 1 - 2p - \delta > p(1 - 2e^{-\varepsilon}) + 1 - 2p - \delta = 1 - p - \delta - 2pe^{-\varepsilon}.\end{aligned} \tag{4.39}$$

Taking into account equation (4.14) and estimates (4.37), (4.39), we get

$$1 > \frac{1 - p - \delta - 2pe^{-\varepsilon}}{\mu - 1} > \beta_c \frac{1 - p - \delta - 2pe^{-\varepsilon}}{\ln(2(1-p)/\delta)},$$

that gives the upper bound:

$$\beta_c < \frac{1}{1 - p - \delta - 2pe^{-\varepsilon}} \ln \frac{2(1-p)}{\delta}. \tag{4.40}$$

4.2.2. Bernoulli Random Potential for the Case $\lambda < +\infty$

We assume in this subsection that $\lambda > \varepsilon + 1$. If the repulsion is very large ($\lambda \gg \varepsilon + 1$), the analysis for $\rho < 1$ is then almost the same as above for $\lambda = +\infty$, whereas for $\rho \geq 1$, which is possible only for finite λ, one needs some more arguments.

Here we start with the estimate the *first-order* correction in λ^{-1} to the value of $\varepsilon_{\rm cr}(\lambda = +\infty) = 2$. With this accuracy the Eqs. (4.5) and (4.6) can be approximated correspondingly by

$$p\left(\frac{\tanh \frac{1}{2}\beta(\mu - \varepsilon - 1)}{\mu - \varepsilon - 1} + \frac{1}{2\lambda + \varepsilon + 1 - \mu}\,\frac{e^{-\beta(1+\varepsilon-\mu)/2}}{\cosh \frac{1}{2}\beta(1 + \varepsilon - \mu)}\right) + (1-p)\left(\frac{\tanh \frac{1}{2}\beta(\mu - 1)}{\mu - 1} + \frac{1}{2\lambda + 1 - \mu}\,\frac{e^{\beta(\mu-1)/2}}{\cosh \frac{1}{2}\beta(\mu - 1)}\right) = 1, \tag{4.41}$$

and by (4.15) as above.

To see this, note that if $\rho < 1$, the dominant contribution in (4.6) must come from the $n = 1$ term, i.e. we must have $h_1 < h_2$, so $\mu < 1 + 2\lambda + \varepsilon$. The other terms in (4.6) are then exponentially small and can be neglected, which leads again to (4.15).

Now, because of the presence of $e^{-\beta h_1}$ in the $n = 2$ term of (4.4), it cannot be neglected in (4.5) and we obtain:

$$\frac{2p}{1 + e^{-\beta(1+\varepsilon-\mu)}}\left\{\frac{e^{-\beta(1+\varepsilon-\mu)} - 1}{\mu - 1 - \varepsilon} + 2\frac{e^{-\beta(1+\varepsilon-\mu)}}{1 + 2\lambda + \varepsilon - \mu}\right\} + \frac{2(1-p)}{1 + e^{-\beta(1-\mu)}}\left\{\frac{e^{-\beta(1-\mu)} - 1}{\mu - 1} + 2\frac{e^{-\beta(1-\mu)}}{1 + 2\lambda - \mu}\right\} = 2,$$

which is the same as (4.41).

Similar to (4.22) the gap equation for $1 < \mu < 1 + \varepsilon$ can be obtained from (4.41) in the limit $\beta \to \infty$:

$$\frac{p}{\varepsilon + 1 - \mu} + (1 - p)\left(\frac{1}{\mu - 1} + \frac{2}{2\lambda + 1 - \mu}\right) = 1. \tag{4.42}$$

If $\rho = 1 - p$, then by (4.15) and (4.29) we again obtain the limit: $\mu \to 1 + \frac{1}{2}\varepsilon$ for $\beta \to \infty$. Inserting this limit into (4.42) we obtain

$$\frac{2}{\varepsilon} + \frac{2(1 - p)}{2\lambda - \frac{1}{2}\varepsilon} = 1. \tag{4.43}$$

Hence, by the reasoning similar to those after (4.29), we obtain the critical value of the Bernoulli random potential $\varepsilon_{\rm cr}(\lambda)$ the expression:

$$\varepsilon_{\rm cr}(\lambda) \approx \frac{2}{1 - (1 - p)/\lambda} = 2 + 2(1 - p)/\lambda + \dots, \tag{4.44}$$

which takes into account that λ is large but *finite*.

Another observation, which is related to the finiteness of λ, concerns the value $\beta_c(\rho = 1)$. For hard-core bosons the arguments in the Sec. 4.2.1 show that this value is *infinite* and the corresponding values of the chemical potential must be greater than $1 + \varepsilon$, see (4.6). Now for finite λ and $\mu > 1 + \varepsilon$ the limit of (4.41), when $\beta \to \infty$, reads as:

$$p\left(\frac{1}{\mu - \varepsilon - 1} + \frac{2}{2\lambda + 1 + \varepsilon - \mu}\right) + (1 - p)\left(\frac{1}{\mu - 1} + \frac{2}{2\lambda + 1 - \mu}\right) = 1. \tag{4.45}$$

If $\rho \geq 1$, then we need to reconsider the density Eq. (4.6), which has the form:

$$\rho = p\,\frac{\sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu-\varepsilon,\lambda)}}{\sum_{n=0}^{\infty} e^{-\beta h_n(\mu-\varepsilon,\lambda)}} + (1 - p)\,\frac{\sum_{n=1}^{\infty} n\, e^{-\beta h_n(\mu,\lambda)}}{\sum_{n=0}^{\infty} e^{-\beta h_n(\mu,\lambda)}}. \tag{4.46}$$

Notice that if $\beta \to +\infty$, then by (4.2) and (4.46) one obtains the following limits: $\rho \to 1$, when $\mu \in (1 + \varepsilon, 1 + 2\lambda)$, $\rho \to 2 - p$, when $\mu \in (1 + 2\lambda, 1 + 2\lambda + \varepsilon)$, and $\rho \to 2$, when $\mu \in (1 + 2\lambda + \varepsilon, 1 + 4\lambda)$.

Therefore, at $\rho = 1$ for large β we can ignore in (4.46) the terms higher than h_2, see (4.2), and write in this limit:

$$\begin{aligned}1 \approx\; & p\left\{\frac{e^{-\beta(1+\varepsilon-\mu)} + 2e^{-2\beta(1+\lambda+\varepsilon-\mu)}}{1 + e^{-\beta(1+\varepsilon-\mu)} + e^{-2\beta(1+\lambda+\varepsilon-\mu)}}\right\}\\ & + (1 - p)\left\{\frac{e^{-\beta(1-\mu)} + 2e^{-2\beta(1+\lambda-\mu)}}{1 + e^{-\beta(1-\mu)} + e^{-2\beta(1+\lambda-\mu)}}\right\}\\ & = p\left\{\frac{1 + 2e^{-\beta(1+2\lambda+\varepsilon-\mu)}}{1 + e^{-\beta(\mu-1-\varepsilon)} + e^{-\beta(1+2\lambda+\varepsilon-\mu)}}\right\}\end{aligned}$$

$$+ (1-p)\left\{\frac{1+2e^{-\beta(1+2\lambda-\mu)}}{1+e^{-\beta(\mu-1)}+e^{-\beta(1+2\lambda-\mu)}}\right\}$$
$$\approx 1 + p\left(e^{-\beta(1+2\lambda+\varepsilon-\mu)} - e^{-\beta(\mu-1-\varepsilon)}\right)$$
$$+ (1-p)\left(e^{-\beta(1+2\lambda-\mu)} - e^{-\beta(\mu-1)}\right). \tag{4.47}$$

This yields

$$e^{2\beta\mu} \approx e^{2\beta(1+\lambda)}\frac{1-p+pe^{\beta\varepsilon}}{1-p+pe^{-\beta\varepsilon}} \approx \frac{p}{1-p}e^{2\beta(1+\lambda+\frac{1}{2}\varepsilon)}.$$

The chemical potential defined by Eq. (4.46) therefore tends (for $\rho = 1$) to $1 + \lambda + \frac{1}{2}\varepsilon$ as $\beta \to +\infty$.

Therefore, inserting this into (4.45) we obtain the estimate for the value of *repulsion* $\lambda_{c,1}$ that ensures that $\beta_c(\rho = 1) = +\infty$ in the presence of the random Bernoulli potential:

$$\lambda_{c,1}(\varepsilon) = \frac{1}{2}\left[3 + \sqrt{9 + 2\varepsilon(1 - 2p + \frac{1}{2}\varepsilon)}\right]. \tag{4.48}$$

Remark 4.1. *In the absence of disorder, i.e. if* $\varepsilon = 0$*, the critical value of* λ *is* $\lambda_{c,1} = 3$ *as opposed to* $\lambda_1 = \frac{1}{2}(3 + \sqrt{8})$ *as suggested in Ref. 4. The reason is the same as above for* $\varepsilon_{\rm cr}$*, namely, the graph of* $\mu(\beta, \rho)$ *at* $\rho = 1$ *tends to* $1 + \lambda$ *as* $\beta \to +\infty$ *and this lies in the gap only if* $\lambda \geq 3$*. Similarly, the next critical values are given by*

$$\lambda_{c,k}(\varepsilon = 0) = 2k + 1. \tag{4.49}$$

Remark 4.2. *In Sec. 4.2.1 we notice a new phenomenon specific for the random case: divergence of* β_c *at* $\rho = 1 - p$ *for hard-core bosons, cf. Fig. 1 for* $p = 1/2$*. Instead of fixing* λ*, fixing* $\varepsilon > 2$ *it follows from (4.43) that there is a critical value of the repulsion* $\lambda_{c,1-p}(\varepsilon)$ *(instead of* ε *as in (4.44)) so that* $\beta_c(\rho = 1 - p)$ *diverges for* $\lambda \geq \lambda_{c,1-p}(\varepsilon)$ *in the presence of the random Bernoulli potential:*

$$\lambda_{c,1-p}(\varepsilon) = \frac{\varepsilon}{4} + \frac{\varepsilon(1-p)}{\varepsilon - 2}. \tag{4.50}$$

This critical value is not evident from Fig. 1 as $\varepsilon = 2$.

Remark 4.3. *In Sec. 4.1 we remarked that the critical temperature for free bosons increases due to disorder. We also remarked that for the interacting case this is a more subtle matter, since it depends on the value of repulsion. For large*

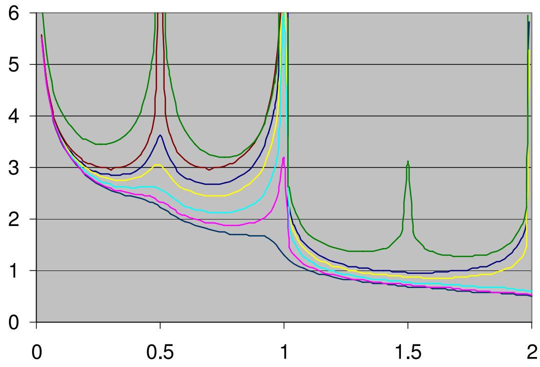

Fig. 1. β_c as a function of the density ρ in the case of averaging over two energies: 0 and $\varepsilon = 2$ with equal probabilities, for various values of λ: $\lambda = 3, 3.3, 4, 6, 10$ and $+\infty$. The top graph corresponds to the case $\varepsilon = 4.5$ and $\lambda = 10$.

repulsions close to e.g. $\lambda_{c,1}(\varepsilon = 0) = 3$*, we get by (4.48) that*

$$\beta_c(\rho = 1; \lambda = 3, \varepsilon > 0) < \beta_c(\rho = 1; \lambda = 3, \varepsilon = 0) = +\infty. \tag{4.51}$$

This lowering of $\beta_c(\rho = 1)$ *can be explained intuitively as follows. At density* $\rho = 1$*, there is one particle per site. If* $\varepsilon = 0$ *there is a penalty for a particle to jump to an already occupied site, so the preferred state is where the particles are at fixed sites, which is almost an eigenstate of the number operators* n_x *for each site. This prevents Bose condensation. (This argument was presented also in Ref. 4.) However, if* $\varepsilon > 0$*, then the lattice splits into two parts with energies 0 and* ε*, and a particle jumping from a site with energy* ε *to a site with energy 0 loses an energy* ε*, which counteracts the gain of* λ*. This creates more freedom of movement and therefore promotes Bose condensation. On the other hand, for a fractional value of the* ρ *in the neighbourhood of* $\rho = 1 - p$*, the critical temperature decreases with increasing* ε *as can be seen from Fig. 1.*

Now consider the case $\rho > 1$. From Eq. (4.46) we see that at fixed $\rho \in (1, 2 - p)$, $\mu \to 1 + 2\lambda$ and for $\rho \in (2 - p, 2)$, $\mu \to 1 + 2\lambda + \varepsilon$ as $\beta \to \infty$.

For the case $\rho = 2 - p$, we have to expand (4.46), as above for $\rho = 1$, see (4.47), but to take into account that $\mu \in (1 + 2\lambda, 1 + 2\lambda + \varepsilon)$:

$$\begin{aligned}
\rho &\approx p\left\{\frac{1 + 2e^{-\beta(1+\,2\lambda+\varepsilon-\mu)}}{1 + e^{-\beta(\mu-1-\varepsilon)} + e^{-\beta(1+2\lambda+\varepsilon-\mu)}}\right\} \\
&\quad + (1-p)\left\{\frac{e^{\beta(1+2\lambda-\mu)} + 2}{1 + e^{\beta(1+2\lambda-\mu)} + e^{-2\beta(\mu-1-\lambda)}}\right\} \\
&\approx 2 - p + p\left(e^{-\beta(1+\varepsilon+2\lambda-\mu)} - e^{-\beta(\mu-1-\varepsilon)}\right) \\
&\quad - (1-p)e^{-\beta(\mu-1-2\lambda)} - 2(1-p)e^{-2\beta(\mu-1-\lambda)}.
\end{aligned} \tag{4.52}$$

This yields that $e^{-\beta(\mu-1-2\lambda)} \approx e^{-\beta(1+\varepsilon+2\lambda-\mu)}p/(1-p)$ for large β, i.e. $\mu \to 1 + 2\lambda + \frac{1}{2}\varepsilon$, if $\rho = 2 - p$ and $\beta \to \infty$.

For $\mu \approx 1 + 2\lambda + \frac{1}{2}\varepsilon$, one has $h_1(\mu-\varepsilon,\lambda) < h_2(\mu-\varepsilon,\lambda)$. So that the p-terms in (4.41) are unchanged, but $h_1(\mu,\lambda) > h_2(\mu,\lambda) < h_3(\mu,\lambda)$, if $\lambda > \varepsilon/4$, which corresponds to our initial hypothesis about the value of repulsion: $\lambda > 1+\varepsilon$. Hence, the $(1-p)$-terms are now dominated for large β by $n = 2$ and (4.41) read as

$$\frac{p}{1+e^{-\beta(1+\varepsilon-\mu)}}\left\{\frac{e^{-\beta(1+\varepsilon-\mu)}-1}{\mu-1-\varepsilon}+2\,\frac{e^{-\beta(1+\varepsilon-\mu)}}{1+2\lambda+\varepsilon-\mu}\right\}$$
$$+\frac{1-p}{e^{-\beta(1-\mu)}+e^{-2\beta(1-\mu+\lambda)}}\left\{2\,\frac{e^{-2\beta(1-\mu+\lambda)}-e^{-\beta(1-\mu)}}{\mu-1-2\lambda}+3\,\frac{e^{-2\beta(1-\mu+\lambda)}}{1+4\lambda-\mu}\right\}\approx 1,$$

In the limit $\beta \to \infty$ we obtain from this relation the gap equation

$$p\left(\frac{1}{\mu-1-\varepsilon}+\frac{2}{1+\varepsilon+2\lambda-\mu}\right)$$
$$+(1-p)\left(\frac{2}{\mu-1-2\lambda}+\frac{3}{1+4\lambda-\mu}\right)=1. \tag{4.53}$$

Inserting $\mu = 1 + 2\lambda + \frac{1}{2}\varepsilon$ into (4.53) leads to

$$\frac{1}{2}\varepsilon^2-(2\lambda-1+2p)\varepsilon+8\lambda=0. \tag{4.54}$$

Solutions of (4.54) are:

$$\varepsilon_{\mathrm{cr},\pm}{}^{(2)}=(2\lambda-1+2p)\pm\sqrt{(2\lambda-1+2p)^2-16\lambda}. \tag{4.55}$$

Hence, there is a solution that for large λ has the form:

$$\varepsilon_{\mathrm{cr}}^{(2)}(\lambda)=4\left(1+\frac{3-2p}{2\lambda}\right)+\ldots, \tag{4.56}$$

or other way around, for a given ε we have:

$$\lambda_{c,\rho=2-p}(\varepsilon)=\frac{2(3-2p)}{(\varepsilon-4)}. \tag{4.57}$$

Clearly, this critical value applies only if $\varepsilon > 4$. The top graph of Figure 1 illustrates this behaviour at $\rho = 1.5$ for $\varepsilon = 4.5$ and $\lambda = 10$.

The critical $\beta_c(\rho)$ for the Bernoulli distribution with $p = 1/2$ and $\varepsilon = 2$ is shown in Figure 1 for a number of values of λ. Notice in particular that $\varepsilon < \varepsilon_{cr}(\lambda)$, see (4.44), for all finite λ, so that $\beta_c(\rho = 1/2) < +\infty$.

Also, for $\lambda = 3.3$, one obtains $\beta_c(\rho = 1) < +\infty$ because $3.3 < \lambda_{c,1}$ $(\varepsilon = 2) = (3+\sqrt{13})/2$, see (4.48).

4.2.3. *Trinomial Distribution:* $\lambda = +\infty$

We also briefly consider the trinomial distribution, taking for simplicity equal probabilities, i.e.

$$\varepsilon^{\omega} = \begin{cases} 0 & \Pr = 1/3 \\ \frac{1}{2}\varepsilon & \Pr = 1/3 \\ \varepsilon & \Pr = 1/3. \end{cases} \tag{4.58}$$

For hard-core bosons, $\lambda = +\infty$, Eq. (4.12) for the critical value of $\beta_c(\rho)$ takes the form:

$$\frac{1}{3}\left[\frac{\tanh\frac{1}{2}\beta(\mu-1)}{\mu-1} + \frac{\tanh\frac{1}{2}\beta\left(\mu-1-\frac{1}{2}\varepsilon\right)}{\mu-1-\frac{1}{2}\varepsilon} + \frac{\tanh\frac{1}{2}\beta(\mu-1-\varepsilon)}{\mu-1-\varepsilon}\right] = 1. \tag{4.59}$$

The density Eq. (4.13) now reads as

$$\rho = \frac{1}{2} + \frac{1}{6}\left(\tanh\frac{1}{2}\beta(\mu-1) + \tanh\frac{1}{2}\beta\left(\mu-1-\frac{1}{2}\varepsilon\right) + \tanh\frac{1}{2}\beta(\mu-1-\varepsilon)\right). \tag{4.60}$$

Then by the same analysis as in Sec. 4.2.1 one gets from (4.60):

$$\lim_{\beta\to\infty}\rho(\beta,\mu) = \begin{cases} 0 & \text{if} \quad \mu < 1 \\ 1/6 & \text{if} \quad \mu = 1 \\ 1/3 & \text{if} \quad 1 < \mu < 1+\varepsilon/2 \\ 1/2 & \text{if} \quad \mu = 1+\varepsilon/2 \\ 2/3 & \text{if} \quad 1+\varepsilon/2 < \mu < 1+\varepsilon \\ 5/6 & \text{if} \quad \mu = 1+\varepsilon \\ 1 & \text{if} \quad \mu > 1+\varepsilon. \end{cases}$$

Alternatively, this can be also expressed as:

$$\lim_{\beta\to\infty}\mu(\beta,\rho) = \begin{cases} 1 & \text{if} \quad 0 < \rho < 1/3; \\ 1+\varepsilon/4 & \text{if} \quad \rho = 1/3; \\ 1+\varepsilon/2 & \text{if} \quad 1/3 < \rho < 2/3; \\ 1+3\varepsilon/4 & \text{if} \quad \rho = 2/3; \\ 1+\varepsilon & \text{if} \quad \rho > 2/3. \end{cases}$$

Again, similar to the reasoning in Sec. 4.2.1, inserting $\mu = 1+\varepsilon/4$ or $\mu = 1+3\varepsilon/4$ into the limiting Eq. (4.59) for $\beta \to +\infty$ yields the critical value of the random potential:

$$\varepsilon_{\mathrm{cr}} = \frac{28}{9}. \tag{4.61}$$

Therefore, (similar to the Bernoulli case for $\rho = 1/2$) the condensation of hard-core bosons is absent at densities $\rho = 1/3$ and $\rho = 2/3$, if $\varepsilon \geq \varepsilon_{\mathrm{cr}}$. This

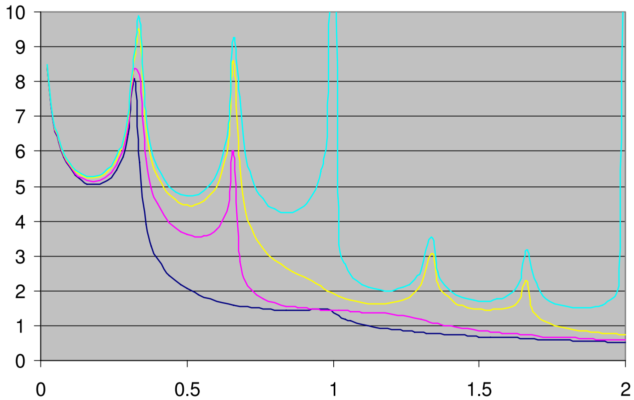

Fig. 2. β_c as a function of the density ρ in the case of a trinomial distribution with width $\varepsilon = 10$ for $\lambda = 3, 4, 6$ and 8.

phenomenon of course persists for $\lambda < +\infty$ and there are similar suppressions of Bose condensation at $\rho = 4/3$, $5/3$, etc., if ε is large enough.

4.2.4. Trinomial Distribution: $\lambda < +\infty$

For $\lambda < +\infty$ there is a similar enhancement of Bose condensation at $\rho = 1$ as for the Bernoulli distribution, but the effect is stronger. This can be seen in Figure 2. The explanation is similar to that in Remark 4.3, except now the lattice splits into 3 equal parts with energies 0, $\varepsilon/2$ and ε. Particles can jump from a singly-occupied site with energy ε to a singly-occupied site with energy 0 or $\varepsilon/2$, thus compensating for the energy penalty of λ due to double occupation.

By Eq. (4.13) for (4.58) we obtain that at $\rho = 1$, $\mu(\beta, \rho) \to 1 + \lambda + \varepsilon/2$ as $\beta \to +\infty$. The gap equation (4.59) then reduces to

$$\frac{1}{\lambda - \varepsilon/2} + \frac{1}{\lambda} + \frac{1}{\lambda + \varepsilon/2} = 1.$$

We can solve it for ε provided $\lambda \geq 3$:

$$\varepsilon_{\rm cr}(\lambda) = 2\lambda\sqrt{\frac{\lambda - 3}{\lambda - 1}}. \tag{4.62}$$

Thus, Bose condensation is absent, if $\lambda \geq 3$ and $\varepsilon \leq \varepsilon_{\rm cr}(\lambda)$.

Figure 2 shows $\beta_c(\rho)$ for a fixed $\varepsilon = 10$ and for values of $\lambda \geq 3$. Then $\varepsilon \geq \varepsilon_{\rm cr}(\lambda = 3, 4, 6)$, but $\varepsilon < \varepsilon_{\rm cr}(\lambda = 8) = 13.52$, which excludes condensation at $\rho = 1$ in the latter case.

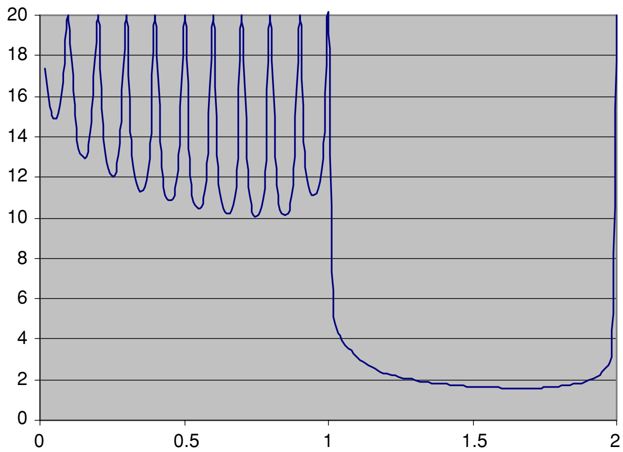

Fig. 3. β_c as a function of the density ρ in the case of averaging over 10 energy values with width $\varepsilon = 10$ for $\lambda = 8$.

4.2.5. General Discrete Distribution

The same phenomena persist for higher numbers of random potential energy values, but the critical value $\varepsilon_{\rm cr}(\lambda)$ becomes rapidly very large. Figure 3 shows the case of a distribution with equal probabilities $\Pr = 1/10$ at 10 equidistant values of ε^ω (with maximal value $\varepsilon = 10$) for $\lambda = 8$. Clearly, condensation is suppressed at $\rho = 1/10, \ldots, 9/10$ and $\rho = 1, 2$ but not at corresponding fractional values above 1, cf. Figure 2.

4.3. Continuous Distribution

4.3.1. The Case $\lambda = +\infty$

Consider *i.i.d.* random potential with *homogeneous* distribution between 0 and ε. In case $\lambda = +\infty$ the Eqs. (4.12) and (4.13) become

$$\frac{1}{\varepsilon}\int_0^\varepsilon \frac{\tanh\frac{1}{2}\beta(\mu-1-x)}{\mu-1-x}\,dx = 1 \tag{4.63}$$

and

$$\frac{1}{\varepsilon}\int_0^\varepsilon \tanh\frac{1}{2}\beta(\mu-1-x)\,dx = 2\rho - 1. \tag{4.64}$$

The latter has sense only for $0 \le \rho \le 1$ and can be solved exactly for μ:

$$\frac{2}{\beta\varepsilon}\ln\frac{e^{\beta(\mu-1)/2}+e^{-\beta(\mu-1)/2}}{e^{\beta(\mu-1-\varepsilon)/2}+e^{-\beta(\mu-1-\varepsilon)/2}} = 2\rho - 1,$$

and hence

$$\mu(\beta,\rho) = 1 + \frac{1}{2}\varepsilon + \frac{1}{\beta}\ln\frac{\sinh\frac{1}{2}\beta\rho\varepsilon}{\sinh\frac{1}{2}\beta(1-\rho)\varepsilon}. \tag{4.65}$$

As $\beta \to +\infty$, the expression (4.65) takes the form

$$\lim_{\beta\to+\infty}\mu(\beta,\rho) := \overline{\mu}(\rho) = 1 + \varepsilon\rho, \quad 0 < \rho < 1, \tag{4.66}$$

whereas $\overline{\mu}(\rho = 0) \in (-\infty, 1]$ and $\overline{\mu}(\rho = 1) \in [1+\varepsilon, +\infty)$ for extreme values of density, i.e., the inverse function is

$$\overline{\rho}(\mu) = \begin{cases} 0 & \mu \le 1 \\ (\mu-1)/\varepsilon & 1 < \mu < 1+\varepsilon \\ 1 & 1+\varepsilon \le \mu. \end{cases} \tag{4.67}$$

Then by (4.63) and (4.67) we obtain for $\rho = 1$ in the limit $\beta \to +\infty$:

$$1 = \frac{1}{\varepsilon}\int_0^{\varepsilon}\frac{1}{\mu - 1 - x}dx,$$

or we get explicitly the value of the chemical potential

$$\overline{\mu}(\rho = 1) = 1 + \frac{\varepsilon}{1 - e^{-\varepsilon}} > 1 + \varepsilon,$$

and similarly

$$\overline{\mu}(\rho = 0) = 1 - \frac{\varepsilon e^{-\varepsilon}}{1 - e^{-\varepsilon}} < 1.$$

Hence, for hard-core bosons the critical $\beta_c(\rho)$ is infinite at extreme densities $\rho = 0, 1$ for *any value* $\varepsilon > 0$ of the uniform continuous distribution.

If $0 < \rho < 1$, then solution of the Eq. (4.64) in the limit $\beta \to +\infty$ is (4.66), whereas the integral in (4.63) diverges. Therefore, if the critical $\beta_c(0 < \rho < 1)$ exist, it must be bounded. Moreover, since $(\tanh u)/u \le 1$, by (4.63) we get for it a bound from below: $2 < \beta_c(0 < \rho < 1)$.

To prove the existence and uniqueness of $\beta_c(0 < \rho < 1)$ consider first (4.64) for $\rho \le \frac{1}{2}$. Then by virtue of (4.65) for *any* finite β the solution $\mu(\beta,\rho)$ increases from $-\infty$ to $1 + \varepsilon/2$ when ρ changes from 0 to 1/2. For this variation of chemical potential the integral in the left-hand side of (4.63) increases monotonously from 0 to its *maximal* value given by

$$I(\beta, \mu = 1 + \varepsilon/2) = \frac{1}{\varepsilon}\int_0^{\varepsilon}\frac{\tanh\frac{1}{2}\beta(x - \varepsilon/2)}{x - \varepsilon/2}dx. \tag{4.68}$$

Indeed,

$$\partial_\mu I(\beta,\mu) = \frac{1}{\varepsilon}\left(\frac{\tanh\frac{1}{2}\beta(\mu-1)}{\mu-1} - \frac{\tanh\frac{1}{2}\beta(\mu-1-\varepsilon)}{\mu-1-\varepsilon}\right) \ge 0$$

for $\mu \leq 1 + \varepsilon/2$. The integral in (4.68) is obviously an increasing function of β. So, there exist $\beta_0 > 2$ such that the maximal value of integral $I(\beta_0, \mu = 1 + \varepsilon/2) \geq 1$. Hence, for any $\beta \geq \beta_0$ there is a *unique* density $0 < \overline{\rho}(\beta) \leq 1/2$ such that

$$I(\beta, \mu(\beta, \overline{\rho}(\beta)) = 1. \tag{4.69}$$

Notice that by (4.65) $\mu(\beta, \rho)$ is increasing of the both arguments: β and $0 < \rho \leq 1/2$. Hence, to satisfy (4.69) $\overline{\rho}(\beta)$ must be decreasing function of β, i.e., the *inverse* function $\beta_c = \beta_c(\rho)$ is also a decreasing with $\lim_{\rho\to 0} \beta_c(\rho) = +\infty$ and $\lim_{\rho\to 1/2} \beta_c(\rho) \geq \beta_0$.

Similar arguments are valid for $1/2 \leq \rho < 1$. Whereas $\mu(\beta, \rho)$ is still increasing function of ρ, the integral $I(\beta, \mu)$ now decreases with μ from its maximal value (4.68) to 0. Therefore, $\beta_c = \beta_c(\rho)$ is a monotonously increasing function of ρ with $\lim_{\rho\to 1/2} \beta_c(\rho) \geq \beta_0$ and $\lim_{\rho\to 1} \beta_c(\rho) = +\infty$, i.e. with a minimum at $\rho = 1/2$ as we have seen for discrete distributions and hard-core bosons.

4.3.2. The Case of Large $\lambda < +\infty$

By virtue of Eqs. (4.5) and (4.6), for $\lambda < +\infty$, the Bose condensate is still suppressed at $\rho = k$.

The analysis is very similar to the case $\varepsilon = 0$. In the limit $\beta \to +\infty$ by (4.6) the density tends to ($k = 0, 1, \ldots$)

$$\rho(\mu, \beta) \to \begin{cases} 0 & \text{if } \mu < 1 \\ k + \frac{1}{\varepsilon}(\mu - 1 - 2k\lambda) & \text{if } 1 + 2k\lambda < \mu < 1 + 2k\lambda + \varepsilon \\ k + 1 & \text{if } 1 + 2k\lambda + \varepsilon < \mu < 1 + 2(k+1)\lambda. \end{cases}$$

(To see this note that if $1 + 2k\lambda < \mu < 1 + 2k\lambda + \varepsilon$ then the term $e^{-\beta h_{k+1}}$ dominates for $x < \mu - 1 - 2k\lambda$ and the term $e^{-\beta h_k}$ dominates for $x > \mu - 1 - 2k\lambda$.) Clearly, if $0 < \rho < 1$ then for solution of (4.6) one gets as above: $\mu(\beta, \rho) \to 1 + \rho\varepsilon$ when $\beta \to +\infty$. If $\rho = 1$, we need to approximate (4.6) more carefully:

$$\begin{aligned} 1 &\approx \frac{1}{\varepsilon} \int_0^{\varepsilon} \frac{e^{\beta(\mu-1-x)} + 2e^{2\beta(\mu-1-x-\lambda)}}{1 + e^{\beta(\mu-1-x)} + e^{2\beta(\mu-1-x-\lambda)}} dx \\ &\approx \frac{1}{\varepsilon} \int_0^{\varepsilon} \left[1 + e^{-\beta(1+x+2\lambda-\mu)} - e^{-\beta(\mu-1-x)}\right] dx. \end{aligned}$$

Working out the integral, we find that $\mu(\beta, \rho = 1) \to 1 + \lambda + \frac{1}{2}\varepsilon$ as $\beta \to +\infty$. More generally, if $\rho = k$, $\mu(\beta, \rho = k) \to 1 + (2k-1)\lambda + \frac{1}{2}\varepsilon$. For large β, the gap Eq. (4.5) becomes

$$\frac{1}{\varepsilon} \int_0^{\varepsilon} \left\{ \frac{k}{\mu - 1 - 2(k-1)\lambda) - x} + \frac{k+1}{1 + 2k\lambda + x + 2\lambda - \mu} \right\} dx = 1.$$

Inserting $\mu = 1 + (2k-1)\lambda + \frac{1}{2}\varepsilon$ we obtain that

$$\frac{1}{\varepsilon}\int_0^\varepsilon \left\{ \frac{k}{\lambda + \frac{1}{2}\varepsilon - x} + \frac{k+1}{\lambda - \frac{1}{2}\varepsilon + x} \right\} dx = 1.$$

This gives for the critical values of repulsion:

$$\lambda_{c,k}(\varepsilon) = \frac{1}{2}\varepsilon\, \frac{e^{\varepsilon/(2k+1)} + 1}{e^{\varepsilon/(2k+1)} - 1}. \tag{4.70}$$

It is easy to see that this is larger than for non-random case $\lambda_{c,k}(0) = 2k+1$ and agrees with the value mentioned above at $\varepsilon = 0$, see Sec. 4.2.2.

Figure 4 shows the phase diagram for $\lambda = 10$ with $\varepsilon = 3$, taking an average over a uniform distribution corresponding to 10 equidistant random values of ε^ω in the interval [0, 3]. It shows that this already approximates the continuous case quite well.

4.3.3. The Case of Small $\lambda > 0$

We finally consider the case of small λ. Figure 5 shows that, in contradistinction to the case $\lambda = 0$, for small λ the critical $\beta_c(\lambda, \varepsilon) > \beta_c(\lambda = 0, \varepsilon = 0)$, i.e. it is *larger* than that at $\varepsilon = 0$!

This can be understood as follows. Whereas in the free case $\lambda = 0$, we must have $\mu < 0$, when $\lambda > 0$, this is no longer so. In the limit $\lambda \to 0$, we can replace $e^{-\beta h_n(\mu,\lambda)}$ in the expression (4.4) for $\tilde{p}''(\beta, \mu, \lambda; 0)$ occurring in the gap Eq. (4.5) by $e^{\beta(\mu-1)}$. Replacing also $h_{n-1} - h_n$ (see (4.2)) by $\mu - 1$ the series (4.4) can be

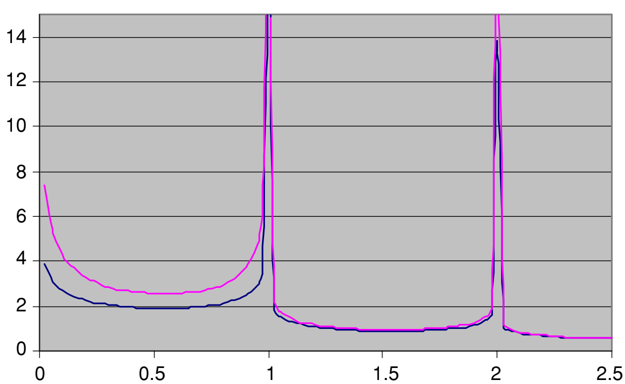

Fig. 4. β_c as a function of the density ρ in the case of a near-continuous distribution: averaging over 10 energy values with width $\varepsilon = 3$ for $\lambda = 10$. The lower graph is the case without randomness.

summed and we obtain for (4.5):

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\frac{1}{1+x-\mu}dx = 1.$$

If $\varepsilon = 0$ this leads to the free gas critical value $\mu = 0$, but for $\varepsilon > 0$ we obtain

$$\mu = \frac{e^{\varepsilon}-1-\varepsilon}{e^{\varepsilon}-1} > 0. \tag{4.71}$$

Similarly, the density Eq. (4.6) now reads as

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\frac{1}{e^{\beta(1-\mu+x)}-1}dx = \rho. \tag{4.72}$$

By (4.71) we can approximate for small ε μ by $\mu \approx \varepsilon/2$ and inserting it in (4.72) we find

$$\frac{1}{\varepsilon}\int_0^{\varepsilon}\frac{1}{e^{\beta(1-\varepsilon/2+x)}-1}dx = \rho. \tag{4.73}$$

By convexity of the function $(e^{\beta(1+x)}-1)^{-1}$, we conclude for solution of the Eq. (4.73) that

$$\beta_c(\rho,\varepsilon) > \beta_c(\rho,0) = \ln\left(1+\frac{1}{\rho}\right).$$

Notice that this argument also applies in the case of a discrete distribution, see Fig. 5.

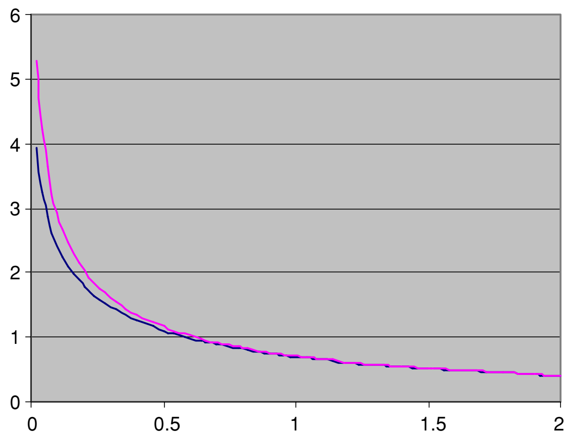

Fig. 5. β_c as a function of the density ρ in the case of averaging over two energies and width $\varepsilon = 2$ for small $\lambda = 0.1$. For comparison, the lower graph shows the case without randomness.

5. CONCLUSION

We conclude with a few remarks concerning our results and open problems. Summarizing the most striking observations about the model considered in this paper, we have seen that at large values of the on-site repulsion with a discrete distribution of the *i.i.d.* random single-site particle potential, the disorder causes a suppression of Bose–Einstein condensation at fractional values of the density. On the other hand, the suppression of Bose–Einstein condensation at integer values of the density observed in the absence of disorder may be lifted. For continuous distributions we found that the critical temperature decreases with increasing disorder for non-integer densities.

We have concentrated here on the case of an *independent identically* and uniformly distributed random external potential. Nonuniform distributions as well as a random on-site interaction may also be of interest and give rise to new phenomena. Of course, all our results concern the infinite-range-hopping model. It would be of considerable interest to extend our results to the short-range hopping model.

ACKNOWLEDGMENTS

This paper is a result of a number mutual visits: TCD would like to acknowledge an enjoyable visit to the Centre de Physique Théorique - Luminy - UMR 6207: Université de Provence - Aix-Marseille I, Université de la Méditerranée - Aix-Marseille II, Université du Sud - Toulon - Var, FRUMAM (FR 2291) and CNRS. LAP and VAZ are thankful to the School of Theoretical Physics for hospitality and support during their visits of Dublin Institute for Advanced Studies.

REFERENCES

1. T. Matsubara and H. Matsuda, A lattice model of liquid helium. *Progr. Theor. Phys.* **16**:569–582 (1956).
2. D. Ueltschi, Geometric and probabilistic aspects of boson lattice models. In *Progr. Probab.* **51**:363–391 (Birkhäuser, Boston, MA, 2002).
3. G. Greiner, O. Mandel, T. Esslinger, T. W. Hänsch and I. Bloch, Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms. *Nature* **415**:39–44 (2002).
4. J.-B. Bru and T. C. Dorlas, Exact solution of the infinite-range-hoping Bose–Hubbard model. *J. Stat. Phys.* **113**:177–195 (2003).
5. M. Aizenman, E. H. Lieb, R. Seilinger, J. P. Solovej and J. Yngvason, Bose–Einstein quantum phase transition in an optical lattice model. *Phys. Rev.* A **70**:023612-1–12 (2004).
6. M. Kac and J. M. Luttinger, Bose–Einstein condensation in the presence of impurities. *J. Math. Phys.* **14**:1626–1628 (1973).
7. M. Kac and J. M. Luttinger, Bose–Einstein condensation in the presence of impurities II. *J. Math. Phys.* **15**:183–186 (1974).

8. J. M. Luttinger and H. K. Sy, Bose–Einstein condensation in a One-Dimentional Model with Random Impurities. *Phys. Rev.* A **7**:712–720 (1973).
9. O. Lenoble, L. A. Pastur and V. A. Zagrebnov, Bose–Einstein condensation in random potentials. *Comptes-Rendus de l'Académie des Sciences* (Physique) **5**: 129–142 (2004).
10. O. Lenoble and V. A. Zagrebnov, Bose–Einstein condensation in the Luttinger-Sy model. (*subm. to Markov Proc. & Rel. Fields*).
11. M. P. A. Fisher, P. B. Weichman, G. Grinstein and D. S. Fisher, Boson localization and the superfluid-insulator transition. *Phys. Rev.* B **40**:456–470 (1989).
12. M. Kobayashi and M. Tsubota, Bose–Einstein condensation and superfluidity of a dilute Bose gas in a random potential. *Phys. Rev.* B **66**:174516-1–7 (2002).
13. W. Krauth, N. Trivedi and D. Ceperley, Superfluid-insulator transition in disordered boson systems. *Phys. Rev. Lett.* **67**:2303–2310 (1991).
14. T. Kennedy, E. H. Lieb and B. S. Shastry, The X-Y model has long-range order for all spins and all dimensions geater than one. *Phys. Rev. Lett.* **61**:2582–2584 (1988).
15. N. Angelescu and M. Bundaru, A remark on the condensation in the hard-core lattice bose gas. *J. Stat. Phys.* **69**:897–903 (1992).
16. N. N. Bogolyubov Jr., J. Brankov, V. A. Zagrebnov, A. M. Kurbatov and N. Tonchev, Some classes of exactly soluble models of problems in Quantum Statistical Mechanics: The method of approximating Hamiltonian. *Russian Math. Surveys* **39**:1–50 (1984).
17. L. A. Pastur and A. L. Figotin, Theory of disordered spin systems. *Theor. Math. Phys.* **35**: 403–414 (1978).
18. L. A. Pastur and M. V. Shcherbina, Infinite-range limit for correlation functions of lattice systems. *Theor. Math. Phys.* **61**:955–964 (1983).
19. M. Reed and B. Simon, *Method of Modern Analysis II: Fourier Analysis, Self-Adjointness* (Academic Press, New York, 1975).
20. V. A. Zagrebnov and J.-B. Bru, The Bogoliubov model of weakly imperfect Bose gas. *Phys. Rep.* **350**:291–434 (2001).
21. E. B. Davis, *One-Parameter Semigroups* (Academic Press, London, 1980).
22. H. Neidhardt and V. A. Zagrebnov, Regularization and convergence for singular perturbations. *Commun. Math. Phys.* **149**:573–586 (1992).
23. M. Reed and B. Simon, *Method of Modern Analysis I: Functional Analysis* (Academic Press, New York, 1972).
24. V. A. Zagrebnov, *Topics in the Theory of Gibbs Semigroups*, Lecture Notes in Mathematical and Theoretical Physics, Vol. 10 (Leuven University Press, Leuven 2003).
25. T. Kato, *Perturbation theory for linear operators*, 2nd ed. (Springer Verlag, Berlin 1980).